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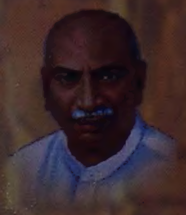

B.Sc., MATHEMATICS

Second Year

Unit : 1 - 10

Paper - III

MECHANICS

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B.SC., Mathematics **MACHANICS**

Second Year **PAPER – III**

Dear Student,

We welcome you as a Student of the Second year B.sc degree Course in mathematics.

This paper – III deals with Mechanics. The learning material for this paper will be supplemented by Contact seminars.

Learning through the Distance Education mode, as you are all aware, involves self – learning and self – assessment and in this regard you are expected to put in disciplined and dedicated effort.

On our part, we assume of our guidance and support. With best wishes.

SYLLABUS

B.SC., Mathematics – Second year – Paper – III

MECHANICS

(Statics and Dynamics)

UNIT – 1: Forces acting at a point – Resultant and components – Parallelogram law of forces – Triangle law of forces – Converse of triangle law of forces – Lami's Theorem.

UNIT – 2: Resolutions of a force – Theorem of resolved parts – Resultant of any number of coplanar forces – Condition of equilibrium.

UNIT – 3: Parallel forces – Resultant of two like and unlike parallel foces – Moment of a force – Varigon's Theorem.

UNIT – 4: Three forces acting on a rigid body.

UNIT – 5: Friction.

UNIT – 6: Projectiles – Path of a projectile – Maximum height – Time taken by a particle – Time of flight – Horizontal Range – Simple Problems.

UNIT – 7: Range on an included plane – Impact – Laws of impact – Impact in a fixed plane.

UNIT – 8: Direct and Oblique impact.

UNIT – 9: Simple Harmonic Motions – Equation of motion – Composition of two S.H.M.'S.

UNIT – 10: Central orbits – Components of velocities and accelerations along and Perpendicular to radius vector – Differential equation of a central orbit – Pedal equations.

Text Books:

1. Statics by M.K. Venkataraman. Agasthiyar Publications. 10th edition July 2002

2. Dynamics by M.K. Venkataraman. Agasthiyar Publications. 11th edition, Feb 2004.

CONTENTS

MECHANICS

(STATICS AND DYNAMICS)

Space for Hints

0.1 Introduction :

Mechanics is the science that deals with the action of forces on bodies. Under the influence of forces, bodies may be either in motion or at rest. That branch of mechanics which is concerned with the conditions under which bodies remain at rest when acted on by forces is called Statics. A study of the motion of a particle or of a rigid body. This part of Mechanics is called Dynamics.

Let us first take, for our study, the first part, namely. Statics and to make such a study, it is necessary to define the terms: Force, Particle, Rigid body and Equilibrium.

0.1.1 FORCE:

A force is that which, acting on a body, changes or has a tendency to change the state of rest or of uniform motion of a body. Associated with the idea of a force we have the following:

(i) a point of application

(ii) a direction and (iii) magnitude.

Since a straight line has both magnitude and direction, a force care be conveniently be represented by a straight line through the point of application.

Thus, the directed line AB, $\overrightarrow{AB}$ represents a force, where A represents the point of application of the force; and the length of AB represents its magnitude.

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0.1.2 PARTICLE:

A Particle is ideally represented by a point and forces on a particle are taken to act at the point which represents the particle, and are necessary concurrent.

Thus to represent forces acting on a particle we need only their magnitudes and directions since the point of application of the forces is the common point taken to represent the particle.

0.1.3 Regid body:

A rigid body is an assemblage of particles such that the distance between any two of them is kept invariant by the internal forces of the system. This concept, though purely ideal, conforms to a fair degree of accuracy, what is observed in practice. Forces acting on a rigid body need not act at the same point, or what is the same as saying that they need not be concurrent. There are cases in which all the forces acting on a rigid body lie in the same plane. The body is then said to be acted by coplanar forces. More generally, the forces may lie in space also. They are then said to constitute forces in three dimensions.

0.1.4 Various types of forces:

i. The force of weight of a body is due to the attraction caused by the Earth This is a vertical force, acting downwards through the centre of gravity of the body.

ii. **Tension of a string:** If a fine string connecting two particles A and B passes over smooth surfaces, The force exerted on the particles by the string is called the force of tension; this force is the same throughout the entire length of the string as long as the string passes over smooth surfaces.

iii. **Thrust on a rod:** If a light rod connects two particles A and B, the force exerted by the rod on both A and B, the force exerted by the rod on both A and B will be the same and is called a thrust. It is said that the force of thrust is negative tension.

iv. Whenever two bodies be in contact with each other, there acts the point of contact, a force of mutual action and reaction between the two bodies. This reaction, unknown in magnitude and direction, permits of a tangential component called the force of friction and a component in the direction of the common normal known as the normal reaction. Where the two bodies are smooth the tangential component is zero and only the normal component exists.

v. **External and internal forces:**

Consider a book kept on a table. The book presses down on the table and the table presses upon the book. If the system is the book and the table taken together, these forces are exerted by the particles of the system and are considered as internal forces. But if the system is the book only, then the force by which the table presses upon the book is, from the point of view of the book an external force. Thus to define: A force acting on a particle of a given system is an "internal" force when it is exerted by another particle of the same system, otherwise it is called an "external force". As a consequence to Newton's third law, it happens that internal forces occur in equal but oppositely directed pairs, each pair constituting the mutual interactions of a pair of particles of the system.

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0.1.5 Equilibrium:

If a number of forces acting on a body keep it at rest, they are said to be in equilibrium.

Two forces acting at a point of a body can be in equilibrium only if they are equal in magnitude and act in opposite directions.

STATICS

UNIT – 1

FORCES ACTING AT A POINT

1.1 Resultant and Components:

Definition: 1.1.1

If two or more forces F_1, F_2, F_3.....etc act on a rigid body and if a single force R can be found whose effect on the body is the same as that of all the forces F_1, F_2, F_3,......etc. Put together, then the single forces R is called the resultant of the forces F_1, F_2, ...etc. and the forces F_1, F_2, F_3,..... etc. are called the components of the force R.

1.1.2 Simple cases of finding the Resultant:

If two forces P and Q act in the same direction simultaneously on a particle, the resultant is clearly equal to a force P + Q acting in the same direction on it. If however P and Q act in opposite directions, their resultant is clearly equal to P~Q and acts in direction of the greater force.

When two forces acting at a point are in different directions (ie.) are inclined to each other their resultant can be found with help of a fundamental theorem in statics known as the law of the parallelogram of Forces.

1.2 Parallelogram law of Forces:

Theorem:

If two forces acting at a point be represented in magnitude and direction, by the sides of a parallelogram drawn from the point their resultant is represented both in magnitude and direction by the diagonal of the parallelogram drawn through that point.

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Proof:

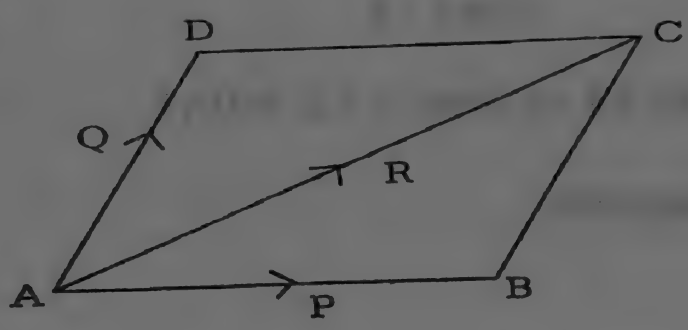

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Fig. (1)

If the two forces P and Q acting at A are represented in magnitude and direction by the straight lines AB and AD if the Parallelogram DAB be completed, then the diagonal AC will represent in magnitude and direction the resultant of P and Q.

In the language of vectors, the above law can be put as

$$\overline{AB} + \overline{AD} = \overline{AC}.$$

1.2.1 Analytical Expression for the Resultant of two forces acting at a point;-

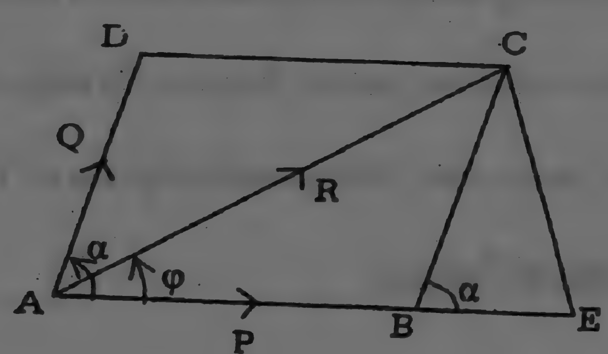

Fig.2

Let the two forces P and Q acting at A be represented by AB and AD let the angle between them be α.

(ie) $\angle BAD = \alpha$.

Complete the parallelogram BAD.

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Then the diagonal AC will represent the resultant. Let R be the magnitude of the result and let it make an angel ϕ with P.

(ie) $\angle CAB = \phi$

Draw CE $\perp$ to AB.

$BC = AD = Q$

From the right angled ΔCBE,

$$\text{Sin}\,\angle CBE = \frac{CE}{BC}$$

$$\text{(ie) Sin}\ \alpha = \frac{CE}{Q}$$

$$\therefore CE = Q\,\text{Sin}\alpha ----(i)$$

$$\text{Cos}\,\alpha = \frac{BE}{BC} = \frac{BE}{Q}$$

$$\therefore BE = Q\,\text{Cos}\,\alpha ----(ii)$$

Now, $R^2 = AC^2 = AE^2 + CE^2 = (AB + BE)^2 + CE^2$

$$= (P + Q\ \text{Cos}\ \alpha)^2 + (Q\,\text{Sin}\,\alpha)^2 = P^2 + Q^2 \cos^2 \alpha + 2PQ\text{Cos}\alpha + Q^2\text{Sin}^2\alpha$$

$$= P^2 + Q^2(\cos^2\alpha + \sin^2\alpha) + 2PQ\cos\alpha$$

$$\therefore R = \sqrt{P^2 + 2PQ\cos\alpha + Q^2} \qquad (1)$$

$$\text{Also}\ \tan\ \phi = \frac{CE}{AE} = \frac{Q\text{Sin}\,\alpha}{P + Q\,\text{Cos}\alpha} \qquad (2)$$

(1) gives the magnitude and (2) the direction of the resultant in terms of P, Q, and α.

Corollary 1:

If the forces P and Q are at right angles to each other, then

$\alpha = 90^0$, $\text{Cos}\,\alpha = \text{Cos}90^0 = 0$ and

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$\operatorname{Sin}\alpha = \operatorname{Sin}90^0 = 1.$

The above results become simpler & we have $R = \sqrt{P^2 + Q^2}$ and $\tan\phi = \frac{Q}{P}$

These results may be easily inferred.

Since the parallelogram becomes a rectangle.

Corollary 2:

If the forces are equal (ie) Q = P, then

$$R = \sqrt{P^2 + 2P^2\cos\alpha + P^2} = \sqrt{2P^2(1+\cos\alpha)}$$

$$= \sqrt{2P^2 2\cos^2\frac{\alpha}{2}} = 2P\cos\frac{\alpha}{2}$$

$$\text{And } \tan\phi = \frac{P\operatorname{Sin}\alpha}{P + P\operatorname{Cos}\alpha} = \frac{\operatorname{Sin}\alpha}{1+\cos\alpha}$$

$$= \frac{2\operatorname{Sin}\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}} = \tan\frac{\alpha}{2}$$

(ie) $\phi = \frac{\alpha}{2.}$

Thus the resultant of two equal forces P, P at an angle α.is$2P\cos\frac{\alpha}{2}$in a direction bisecting the angle between them.

This fact $\left(\text{that}\,\phi = \frac{\alpha}{2}\right)$ is obvious otherwise, as the parallelogram becomes a rhombus.

Corollary: 3

Let the magnitudes P and Q of two forces acting at an angle α be given.

Then their resultant R is greatest when cos α is greatest.

(ie) when $\cos\alpha = 1$ or $\alpha = 0^0$

In this case, the forces act along the same line in the same direction

and $R = P + Q$

The least value of R occurs when Cos α is least.

(ie) When cos α = -1 or $\alpha = 180^0$

In this case, the forces act along the same line but in opposite

directions and $R = P \sim Q$.

Example: - 1

The resultant of two forces P, Q acting at a certain angle X and that of P, R acting at the same angle is also X. The resultant of Q, R again acting at the same angle is Y. Prove that

$$P = (x^2 + QR)^{½} = \frac{QR(Q+R)}{Q^2+R^2-Y^2}$$

Prove also that, if $P + Q + R = 0$, $Y = X$.

Solution:-

Let P and Q act an angle α. From the given data.

We have the following results

$$X^2 = P^2 + Q^2 + 2PQ \text{ Cos}\,\alpha. \quad (1)$$

$$X^2 = P^2 + R^2 + 2PR \text{ Cos}\,\alpha. \quad (2)$$

And $$Y^2 = Q^2 + R^2 + 2QR \text{ Cos}\,\alpha. \quad (3)$$

(1) – (2) gives $0 = Q^2 - R^2 + 2P \text{ Cos}\,\alpha.(Q - R)$

(ie) $0 = (Q - R)\,(Q + R + 2P \text{ Cos}\,\alpha.)$

But $Q \neq R$ and So $Q - R$ is $\neq 0$.

$\therefore\; Q + R + 2P \text{ Cos}\,\alpha = 0.$

(or) $$\text{Cos}\,\alpha = (-)\frac{Q+R}{2P} \quad (4)$$

Substituting (4) in (1)

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We have

$$X^2 = P^2 + Q^2 + 2PQ\left\{-\left(\frac{Q+R}{2P}\right)\right\} = P^2 + Q^2 - Q^2 - QR$$

(or) $P^2 = X^2 + QR$, (ie) $P = (X^2 + QR)^{1/2}$

Substituting (4) in (3)

We have $Y^2 = Q^2 + R^2 + 2QR\left\{-\left(\frac{Q+R}{2P}\right)\right\}$

$$= Q^2 + R^2 - \frac{QR(Q+R)}{2P}$$

$$\therefore \frac{QR(Q+R)}{P} = Q^2 + R^2 - Y^2$$

$$P = \frac{QR(Q+R)}{Q^2+R^2-Y^2}$$

If $P + Q + R = 0$. Then $Q + R = -P$

$\therefore$ From (4), $\cos\alpha = -\left\{\frac{Q+R}{2P}\right\} = \frac{P}{2P} = \frac{1}{2}$

Putting $\cos\alpha = ½$ in (2) and (3)

We have $X^2 = P^2 + R^2 + PR$ (5)

and $Y^2 = Q^2 + R^2 + QR$ (6)

(5) – (6) gives

$$X^2 - Y^2 = P^2 - Q^2 + PR - QR$$

$$= (P-Q)(P+Q) + (P-Q)R$$

$$= (P-Q)(P+Q+R)$$

$$= (P-Q).0 \quad [\because P+Q+R=0]$$

$$X^2 - Y^2 = 0$$

$$\therefore X = Y$$

Example: 2

If the resultant R of two forces P and Q inclined to one another at any given angle makes an angle ϕ with the direction of P, show that the resultant of forces (P +R) and Q acting at the same angle will make an angle $\phi/2$ with the direction of P + R.

Solution: First method:

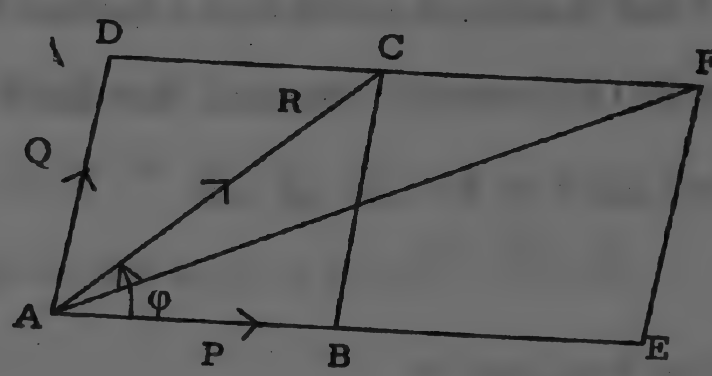

Fig.(3)

Let $\overline{AB} = P$ and $\overline{AD} = Q$

From ||gm ABCD

$$\overline{AB} + \overline{AD} = \overline{AC} = R.$$

To mark the force P + R, Produce AB to E so that BE = AC.

In the || gm DAEF

$\overline{AF}$ gives the new resultant.

In $\Delta CAF, CA = CF$ (each representing R in magnitude)

$$\therefore \angle CAF = \angle CFA$$

$$= \angle FAE \text{ (alternate angles)}$$

(ie) AF bisects $\angle CAB$

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Second method:

The resultant of P + R and Q can be found in two stages. First the resultant of P along AB and Q along AD is a force R along AC. Secondly; We have to find the resultant of the force R along AC with an extra force R along AB. As there are equal, the final resultant bisects the angle BAC.

Example: 3

Two forces P and Q acting at a point have a resultant R. If Q is doubled, R is doubled. Again if Q be reversed in direction, then also R is doubled. Show that $P^2: Q^2: R^2 = 2:3:2$ (or) $P: Q: R = \sqrt{2}:\sqrt{3}:\sqrt{2}$

Solution:

Let P and Q act at an angle α

This resultant R is given by

$$R^2 = P^2 + Q^2 + 2PQ \text{ Cos } \alpha \qquad (1)$$

When Q is doubled (so that the angle between the 2Q and P is again α), R is doubled.

Hence $(2R)^2 = P^2 + (2Q)^2 + 2P.2Q \cos \alpha$

$$\text{Or } 4R^2 = P^2 + 4Q^2 + 4PQ \text{ Cos } \alpha \qquad (2)$$

When Q is reversed. [Angle between the orginal forces now changes to $(180^0 - \alpha)$], R is doubled.

Hence $(2R)^2 = P^2 + Q^2 + 2PQ \text{ Cos } (180^0 - \alpha)$

$$\text{(or) } 4R^2 = P^2 + Q^2 - 2PQ \cos \alpha \qquad (3)$$

Adding (1) & (3)

We get $5R^2 = 2P^2 + 2Q^2$ (4)

Multiplying (3) by 2 and adding with (2)

We get

$12R^2 = 3p^2 + 6Q^2$

Or $4R = P^2 + 2Q^2$ (5)

From (4) & (5)

We obtain on subtraction $R^2 = P^2$.

Substitution for R in (5) gives

$4P^2 = P^2 + 2Q^2$

$3P^2 = 2Q^2$

(ie) $Q^2 = 3/2\ P^2$

Thus $P^2 : Q^2 : P^2 : 3/2\ P^2 : P^2$

$P^2 : Q^2 : R^2 = 2:3:2 \Rightarrow P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}.$

Example: 4

Two forces of magnitudes P Cos A, P Cos B act along the sides CA, CB of a triangle ABC. Prove that their resultant is P sin C and that it makes an angle $90^0 - B$ with CA.

Solution:

The angle between the forces P cos A, P cos B along CA and CB in C.

Hence their resultant R is given by

$R^2 = (P \text{ Cos } A)^2 + (P \text{ Cos } B)^2 + 2(P \text{ Cos } A)(P \text{ Cos } B) \text{ Cos } C$

$= P^2 [\text{Cos}^2 A + \text{Cos}^2 B + 2\text{Cos } A \cos B \cos C]$

But $\text{Cos}^2 A + \text{Cos}^2 B + \text{Cos}^2 C = 1 - 2 \cos A \cos B \cos C$.

Hence $R^2 = P^2 [1 - \text{Cos}^2 C] = P^2 \text{ Sin}^2 C$

Or $R = P \sin C$.

If ϕ is the angle made by the resultant with CA, then

$$\tan\ \phi = \frac{P \text{ Cos } B \text{ Sin } C}{P \text{ Cos } A + P \text{ Cos } B \text{ Cos } C}$$

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$$\tan \phi = \frac{\cos B \sin C}{\cos A + \cos B \cos C}$$

But $\cos A = \cos (180^0 - B - C) = -\cos (B + C)$

$\cos A = -[\cos B \cos C - \sin B \sin C]$

Or $\cos A + \cos B \cos C = \sin B \sin C$

$$\text{Hence } \tan \phi = \frac{\cos B \sin C}{\sin B \sin C} = \cot B$$

$$= \tan (90 - B)$$

$\therefore \phi = 90^0 - B.$

Example: 5

Two forces P + Q and P – Q make an angle 2α with one another and their resultant makes an angle θ with the bisector of the angle between them. Show that $P \tan \theta = Q \tan \alpha$.

Solution:

Since the resultant makes an angle θ with the bisector of the angle between the forces, we find it makes $\alpha - \theta$ with the first force

$$\text{Hence } \tan (\alpha - \theta) = \frac{(P-Q)\sin 2\alpha}{(P+Q)+(P-Q)\cos 2\alpha}$$

$$= \frac{(P-Q)\sin 2\alpha}{P(1+\cos 2\alpha)+Q(1-\cos 2\alpha)}$$

$$= \frac{(P-Q)\, 2\sin 2\alpha \cos \alpha}{P(2\cos^2\alpha)+Q(2\sin^2\alpha)}$$

(ie) $\dfrac{\tan\alpha - \tan\theta}{1+\tan\alpha\tan\theta} = \dfrac{(P-Q)\tan\alpha}{P+Q\tan^2\alpha}$

Cross multiplying and simplifying we get

$$Q \tan \alpha (1+\tan^2\alpha) = P \tan\theta (1+\tan^2\alpha)$$

Since $1 + \tan^2 \alpha$ is non – zero, We get on division by $1 + \tan^2 \alpha$.

$Q \tan \alpha = P \tan \theta$

Example: 6

The greatest resultant that two forces can have is of magnitude P and the least is of magnitude Q. Show that, when they act at an angle α, their resultant is of magnitude $\sqrt{P^2 \cos^2 \frac{1}{2}\alpha + Q^2 \sin^2 \frac{1}{2}\alpha}$.

Solution:

If F_1 and F_2 are two forces, then their greatest resultant is obtained when they are in the same direction through a point.

Hence $P = F_1 + F_2$. (1)

Their resultant is least, if they act in opposite directions through a point.

(ie) $Q = F_1 - F_2$ (2)

Suppose now F_1 and F_2 act through a point at an angle α.

Then their resultant R is given by

$$R^2 = F_1^2 + F_2^2 + 2F_1 . F_2 \cos \alpha \quad (3)$$

Consider

$P^2 \cos^2 \frac{1}{2} \alpha + Q^2 \sin^2 \frac{1}{2} \alpha$

$= (F1 + F_2)^2 \cos^2 \frac{1}{2} \alpha + (F_1 - F_2) \sin^2 \frac{1}{2} \alpha$ [$\therefore$ by (1) & (2)]

$$= F_1^2\left(\cos^2 \frac{1}{2}\alpha + \sin^2 \frac{1}{2}\alpha\right) + F_2^2\left(\cos^2 \frac{1}{2}\alpha + \sin^2 \frac{1}{2}\alpha\right)$$

$$+ 2F_1 F_2 \left(\cos^2 \frac{1}{2}\alpha - \sin^2 \frac{1}{2}\alpha\right)$$

$$= F_1^2 + F_2^2 + 2F_1 . F_2 \cos\alpha$$

$= R^2$ by (3)

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Hence $R = \sqrt{P^2 \cos^2 \frac{1}{2}\alpha + Q^2 \sin^2 \frac{1}{2}\alpha}.$

Example: 7

The resultant of two forces P and Q acting at an angle θ is equal to $(2\ m+1)\sqrt{p^2+Q^2}$; When they act at an angle $90^0 - \theta$, the resultant is $(2m-1)\ \sqrt{p^2}+Q^2$; Prove that $\tan\ \theta = \frac{m-1}{m+1}$.

Solution:

Let θ be the angle between the forces.

$$R = \sqrt{p^2 + Q^2 + 2PQ\cos\theta} \quad \text{--------------} \quad (1)$$

Given that $R = (2m+1)\ \sqrt{p^2+Q^2}$ -------------- (2)

From (1) and (2) we set

$$\Rightarrow (2m+1)^2\ \sqrt{p^2+Q^2} = \sqrt{P^2+Q^2+2\,pQ\cos\theta}$$

Squaring on the both sides we set

$$(2m+1)^2\,(P^2+Q^2) = P^2 + Q^2 + 2\ PQ\ \text{Cos}\,\theta$$

$$\Rightarrow [4m^2+4m+1][p^2+Q^2] = P^2+Q^2+2PQ\,\text{Cos}\,\theta$$

$$\Rightarrow [4m^2+4m+1][p^2+Q^2] - (P^2+Q^2) = 2PQ\ \text{Cos}\,\theta$$

$$\Rightarrow (P^2+Q^2)[4m^2+4m+1-1)=]=2PQ\,\text{Cos}\theta$$

$$\Rightarrow (P^2+Q^2)(4m^2+4m) = 2PQ\,\text{Cos}\theta$$

$$\Rightarrow (P^2+Q^2)\,4m\,(m+1) = 2\,PQ\,\text{Cos}\,\theta \quad \text{----------------} \quad (3)$$

When the forces act at an angle $(90^0 - \theta)$,

$$(2m-1)\ \sqrt{P^2+Q^2} = \sqrt{P^2+Q^2+2PQ\,\text{Cos}\,(90-\theta)}$$

Squaring on both sides we set.

$$(2m-1)^2\,(P^2+Q^2) = P^2+Q^2+2\ PQ\ \text{Sin}\ \theta$$

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$$\Rightarrow (4m^2-4m+1)(P^2+Q^2)=P^2+Q^2+2PQ\sin\theta$$

$$\Rightarrow (4m^2-4m+1)(P^2+Q^2)-(P^2+Q^2)=P^2+Q^2+2PQ\sin\theta$$

$$\Rightarrow (P^2+Q^2)[4m^2-4m+1-1]=2PQ\sin\theta$$

$$\Rightarrow (P^2+Q^2)[4m^2-4m]=2PQ\sin\theta$$

$$\Rightarrow (P^2+Q^2)[4m(m-1]=2PQ\sin\theta \quad \text{--------------(4)}$$

From $\frac{(4)}{(3)} \Rightarrow \frac{(P^2+Q^2)4m(m-1)}{(P^2+Q^2)4m(m+1)} = \frac{2PQ\sin\theta}{2PQ\cos\theta}$

$$\Rightarrow \frac{m-1}{m+1} = \tan\theta$$

$$\therefore \tan\theta = \frac{m-1}{m+1}$$

Example:8

The resultant of two forces P and Q is at right angles to P. Show that the angle between the forces is $\cos^{-1}\left(\frac{-P}{Q}\right)$

Solution:

Let the angle between the forces be θ.

$$\therefore \theta \tan 90^0 = \frac{Q\sin\theta}{P+Q\cos\theta}$$

$$\Rightarrow \quad \alpha = \frac{Q\sin\theta}{P+Q\cos\theta}$$

$$\Rightarrow \frac{1}{0} = \frac{Q\sin\theta}{P+Q\cos\theta}$$

$$\Rightarrow P+Q\cos\theta=0$$

$$\Rightarrow \cos\theta = \frac{-P}{Q}$$

$$\Rightarrow \theta=\cos^{-1}\left(\frac{-P}{Q}\right)$$

Space for Hints

Example:9

The resultant of two forces P and Q is of magnitude P. Show that, if P be doubled, the new resultant is at right angles to Q and its magnitude will be $\sqrt{4P^2 - Q^2}$.

Solution:

Let θ be the angle between the forces P and Q,

then $R^2 = P^2 + Q^2 + 2PQ \cos\theta$

Given that R = P

$$\Rightarrow P^2 = P^2 + Q^2 + 2PQ\cos\theta$$

$$\Rightarrow Q^2 + 2PQ\cos\theta = 0$$

$$\Rightarrow \cos\theta = \frac{-Q^2}{2PQ}$$

$$\Rightarrow \cos\theta = \frac{-Q}{2P} \qquad ------------(1)$$

When P is doubled,

(ie) When P becomes 2P, ℓet the new resultant R' make an angle α with Q, then

$$\tan\alpha = \frac{2P\sin\theta}{Q+2P\cos\theta} \qquad \left[\because \tan\alpha = \frac{P\sin\theta}{Q+P\cos\theta}\right]$$

$$= \frac{2P\sin\theta}{Q+2P.\left(\frac{-Q}{2p}\right)} \qquad [\because \text{by}(1)]$$

$$\tan\alpha = \frac{2P\sin\theta}{Q-Q}$$

$$\tan\alpha = \infty$$

$$\alpha = \frac{\pi}{2}$$

$\therefore$ The new resultant is at right angles to the direction of Q.

$$R'^2 = (2P)^2 + Q^2 + 2(2P)\,Q \cos\theta$$

$$R'^2 = 4P^2 + Q^2 + 4PQ \cos\theta$$

$$= 4P^2 + Q^2 + 4PQ\left(\frac{-Q}{2P}\right) \qquad (\because \text{by } (1))$$

$$= 4P^2 + Q^2 - 2Q^2$$

$$R'^2 = 4P^2 - Q^2$$

$$\Rightarrow R' = \sqrt{4P^2 - Q^2}.$$

Example:10

The greatest and least magnitudes of the resultant of two forces of constant magnitudes are R and S respectively. Prove that, when the forces act an angle 2 ϕ, the resultant is of the magnitude $\sqrt{R^2 \cos^2\phi + S^2 \sin^2\phi}$

Solution:

We know that, when the forces P and Q act at a point making angle α between them.

The resultant is $R^2 = P^2 + Q^2 + 2PQ \cos\alpha$ ---------- (1)

$\therefore$ Maximum of P and Q is give by P + Q

And Minimum of P and Q is given by P – Q (P>Q)

Consider X as the resultant of (P + Q) and (P – Q)

$\therefore$ R = P + Q (Say)

S = P – Q (Say)

$$\Rightarrow P = \frac{R+S}{2};\ Q = \frac{R-S}{2}$$

Now $R^2 = P^2 + Q^2 + 2PQ \cos 2\phi$ $(\because \text{Given that } \theta = 2\phi)$

Space for Hints

Space for Hints

$$X^2 = \left(\frac{R+S}{2}\right)^2 + \left(\frac{R-S}{2}\right)^2 + 2\left(\frac{R+S}{2}\right)\left(\frac{R-S}{2}\right)\text{Cos}\,2\phi$$

$$= \frac{R^2+S^2+2RS}{4} + \frac{R^2+S^2-2RS}{4} + 2\left(\frac{R^2-S^2}{4}\right)\text{Cos}\,2\phi$$

$$= \frac{2(R^2+S^2)+2(R^2-S^2)\text{Cos}2\phi}{4}$$

$$= \frac{1}{2}\left[R^2+S^2+R^2\cos 2\phi - S^2\cos 2\phi\right]$$

$$= \frac{1}{2}[R^2(1+\cos 2\phi)+S^2(1-\cos 2\phi)]$$

$$= \frac{1}{2}[R^2(2\cos^2\phi)+S^2(2\text{Sin}^2\phi)]$$

$$X^2 = R^2\,\text{Cos}^2\,\phi + S^2\,\text{Sin}^2\,\phi$$

$$X = \sqrt{R^2\,\text{Cos}^2\,\phi + S^2\,\text{Sin}^2\,\phi}$$

Exercises

1. Two forces of given magnitudes P and Q act at a Point at an angle α. What will be i) the maximum ii) minimum value of the resultant?

2. Two equal forces act on a particle, find the angle between them when the square of their resultant is equal to three times their product.

3. If the resultant of forces 3P, 5P is equal to 7P find.

i) the angle between the forces.

ii) the angle which the resultant makes with the first force.

4. Two equal forces are inclined at an angle 2θ. Their resultant is 3times as great as when they are inclined at an angle 2ϕ. Show that $\text{Cos}\,\theta = 3\text{Cos}2\phi$.

5. Two forces of magnitudes K cos A and K cos B act along CA, CB of a ΔABC. Prove that their resultant is K sin C.

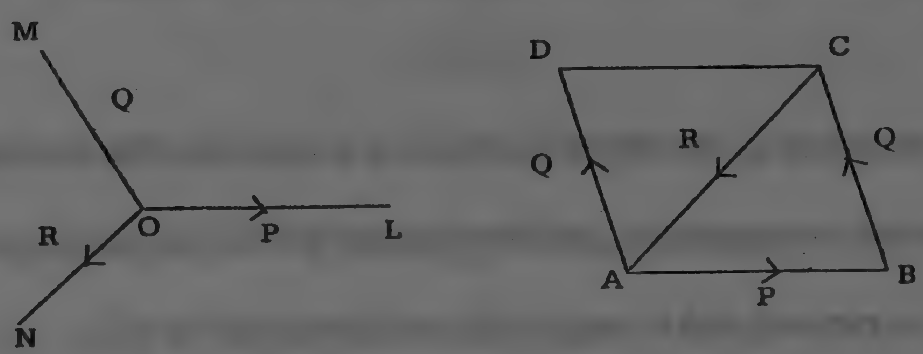
1.3 Triangle law of forces:

A simple deduction from the parallelogram of forces is the following theorem. Known as the Triangle of forces.

Statement:

If three forces acting at a point can be represented is magnitude and direction by the sides of a triangle taken in order, they will be in equilibrium.

Proof:



(a) (b)

Fig.(4)

Let the forces P, Q, R act at a point O and be represented in magnitude and direction by the sides AB, BC, CA of the triangle ABC. We have to prove that they will be in equilibrium.

Complete the parallelogram BADC. As AD is equal and parallel to BC, AD also represents Q in magnitude and direction.

$P + Q = \overline{AB} + \overline{AD}$

$= \overline{AC}$ (by|| gm law).

This shows that the resultant of the forces P and Q at O is represented in magnitude and direction by AC.

The third force R acts at O and it is represented in magnitude and direction by CA.

Space for Hints

Space for Hints

Hence $P + Q + R = \overline{AC}$ at $O + \overline{CA}$ at O

$= \overline{O}$ (as the two vectors at 0 are equal and opposite)

$\therefore$ The forces are in equilibrium.

Note:

In the above theorem, the forces P, Q, R are represented by the sides of the triangle A. B, C only in magnitude and direction but not in position. The forces act at a point and do not act along the sides of the triangle.

Corollary:

From the proof of the above theorem, it is clear that the resultant of the forces represented in magnitude and direction by the two sides AB and BC of the triangle ABC, is represented in magnitude and direction by AC.

This principle is stated as follows:

If two forces acting at a point are represented in magnitude and direction by two sides of a triangle taken in the same order, the resultant will be represented in magnitude and direction by the third side taken in the reverse order.

In the notation of vector, the above means that $\overline{AB} + \overline{BC} = \overline{AC}$

1.3.1 Perpendicular Triangle of Forces:-

Statement:

If three forces acting at a point are such that their magnitudes are proportional to the sides of a triangle and their directions are perpendicular to the corresponding sides, all inwards or all outwards, then also the forces will be in equilibrium.

Proof:

Let the forces P, Q, R meet at O.

Space for Hints

ABC is a triangle such that magnitudes of P, Q, and R are proportional to the sides BC, CA, and AB respectively of ΔABC and their directions are perpendicular to the corresponding sides all outwards.

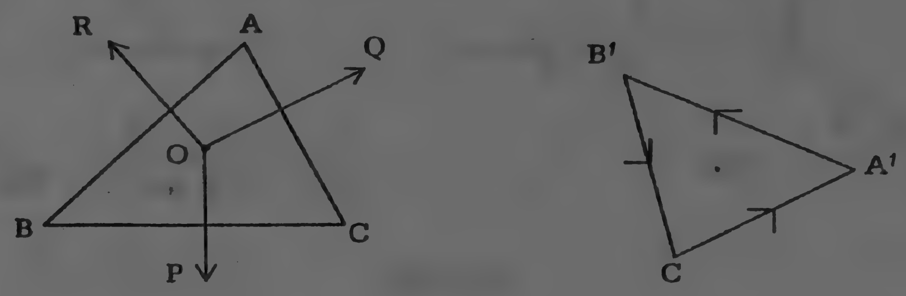


Fig.(5)

We have to prove that they will be in equilibrium.

If the rotate the Δ ABC through 90^0 in its own plane, we will get a new triangle A', B', C' whose sides are parallel to the given forces and represent the forces both in magnitude and direction. Hence by the triangle of forces P. Q, R are in equilibrium.

Note:

The above result will also be true, if the directions of the forces, instead of being perpendicular to the corresponding sides, make equal angles in the same sense with them. The proof is exactly similar.

1.4 Converse of the Triangle of Forces:

Statement:

If three forces acting at a point are in equilibrium, then any triangle drawn so as to have its sides parallel to the directions of the forces shall represent them in magnitude also.

Space for Hints

Proof:

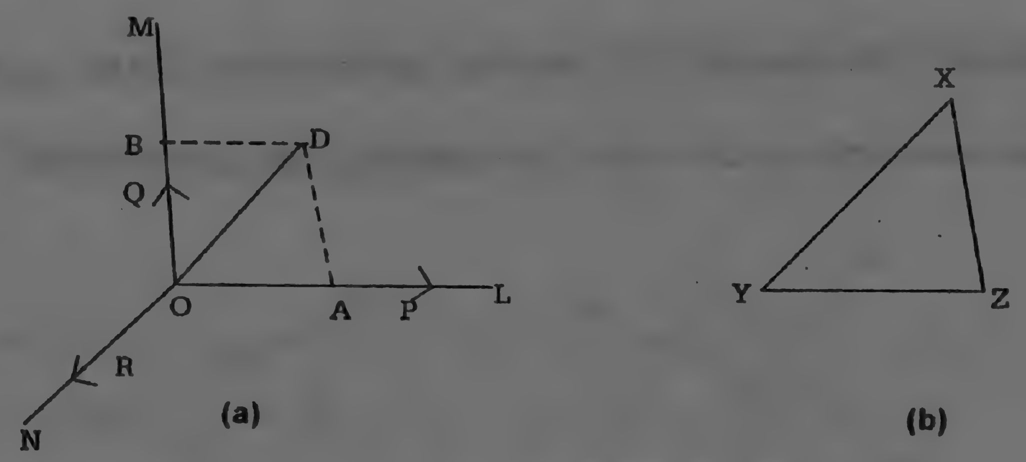


Fig.(6)

Let the three forces P. Q. R acting at O along the directions OL, OM and ON keep it is equilibrium XYZ is a triangle such that the sides YZ, ZX and XY are parallel to the directions of P, Q, R respectively.

We have to prove that to the sides of ΔXYZ are proportional to the magnitudes of P, Q and R given that P + Q + R = O (Statically).

Along OL, Cut off OA to represent the magnitude of P on some scale.

(ie) Let $\overline{OA} = P$.

On the same scale, make $\overline{OB} = Q$.

To get the resultant of P and Q.

Complete the || gm AOB.

The $P + Q = \overline{OA} + \overline{OB} = \overline{OD}$

But $P + Q + R = \overline{O}$ (given)

(ie) $\overline{OD} + R = \overline{O}$ or $R = \overline{DO}$

This shows that the third forces R is represented in magnitude on the same scale by DO and that DON is a straight line.

Hence the three forces P, Q and R are Parallel and proportional to the sides of the sides of the triangle OAD.

Space for Hints

Now any triangle like XYZ whose sides are parallel to the directions of P, Q and R will be similar to ΔOAD and hence

$$\frac{YZ}{OA}=\frac{ZX}{AD}=\frac{XY}{DO}$$

But $$\frac{P}{OA}=\frac{Q}{OB}=\frac{R}{DO}$$

$$\therefore \frac{YZ}{P}=\frac{ZX}{Q}=\frac{XY}{R}$$

(ie) The sides of ΔXYZ will be proportional to P,Q,R.

1.4.1 Polygon of Forces:-

Statement:

If any number of forces acting at a point can be represented in magnitude and direction by the sides of a polygon taken in order, the forces will be in equilibrium.

Proof:

Let the forces P_1, P_2 ......P_n acting at O be represented in magnitude and direction by the sides B_1B_2, B_2B_3......B_nB_1 of the polygon $B_1,B_2....B_n$.

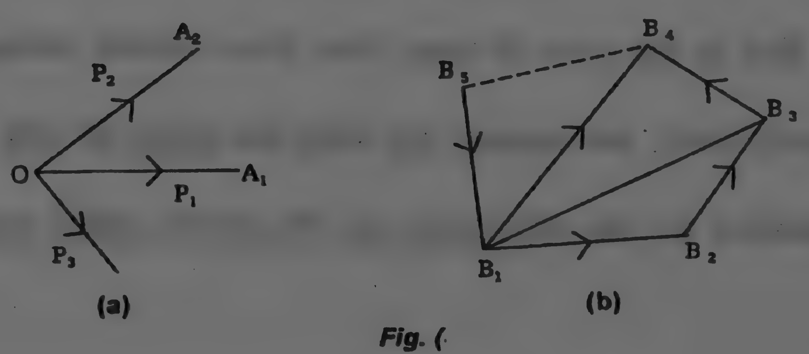


Fig.(7)

We have to prove that the forces will be equilibrium.

Compounding the forces by vector law, step by step.

We have $P_1 + P_2 = \overline{B_1B_2} + \overline{B_2B_3} = \overline{B_1B_3}$

Space for Hints

$P_1 + P_2 + P_3 \quad = \overline{B_1 B_3} + \overline{B_3 B_4} = \overline{B_1 B_4}$

and $P_1 + P_2 + P_3 + \ldots\ldots\ldots + P_{n-1} = \overline{B_1 B_{n-1}} + \overline{B_{n-1} B_n} = \overline{B_1 B_n}$

It is to be noted that in each of the equations above, the resultant on the right side of the forces named on the left side, acts at the point O.

The last force P_n is represented by $\overline{B_n B_1}$.

$\therefore P_1 + P_2 + \ldots\ldots\ldots + P_{n-1} + P_n = \overline{B_1 B_n} \text{ at } O + \overline{B_n B_1} \text{ at } O$

$= \overline{O}.$

$\therefore$ The forces are in equilibrium.

Note 1:

The above theorem is true even when the forces acting at O are not in the same plane.

Note 2:

The converse of the polygon of Forces is not true. The converse of the triangle of forces is true because when ever the directions of three forces acting at a point and keeping it is equilibrium are known, all triangles drawn with their sides parallel to these directions, will be similar and hence represent the forces in magnitude also. But in the case of more than three forces acting at a point and keeping it in equilibrium, we cannot say that the sides of any polygon drawn with its sides parallel to the directions of the forces shall represent them in magnitude also.

It we draw two such polygons, they will be merely equiangular and not necessarily similar. All that we can say is that a polygon can be drawn with the sides parallel and proportional to the forces.

1.5 Lami's Theorem:

Statement:

If three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle between the other two.

Proof:

Refer to the result and figure (6)

We have provided that the sides of the triangle OAD represent the forces P,Q,R in magnitude and direction.

Applying the sine rule to ΔOAD.

We have

$$\frac{OA}{\operatorname{Sin}\angle ODA}=\frac{AD}{\operatorname{Sin}\angle DOA}=\frac{DO}{\operatorname{Sin}\angle OAD}$$

(1)

But $\angle ODA$ = alt. $\angle BOD = 180^0 - \angle MON$

$\therefore \operatorname{Sin}\angle ODA=\operatorname{Sin}(180^0-\angle MON)=\operatorname{Sin}\angle MON$

(2)

Also $\angle DOA = 180^0 - \angle NOL$

$\therefore \operatorname{Sin}\angle DOA = \operatorname{Sin}(180^0 - \angle NOL) = \operatorname{Sin}\angle NOL$

(3)

& $\angle OAD = 180^0 - \angle BOD = 180^0 - \angle LOM$

$\therefore \operatorname{Sin}\angle OAD=\operatorname{Sin}(180^0-\angle LOM)=\operatorname{Sin}\angle LOM$

(4)

Substituting (2), (3), (4) in (1)

We have

$$\frac{OA}{\operatorname{Sin}\angle MON}=\frac{AD}{\operatorname{Sin}\angle NOL}=\frac{DO}{\operatorname{Sin}\angle LOM}$$

Space for Hints

Space for Hints

(ie) $$\frac{P}{\operatorname{Sin}\angle MON}=\frac{Q}{\operatorname{Sin}\angle NOL}=\frac{R}{\operatorname{Sin}\angle LOM}$$

(or) $$\frac{P}{\operatorname{Sin}(Q,R)}=\frac{Q}{\operatorname{Sin}(R,P)}=\frac{R}{\operatorname{Sin}(P,Q)}$$

<u>Example: 1</u>

Two forces act on a particle. If the sum and difference of the forces are at right angles to each other. Show that the forces are of equal magnitude.

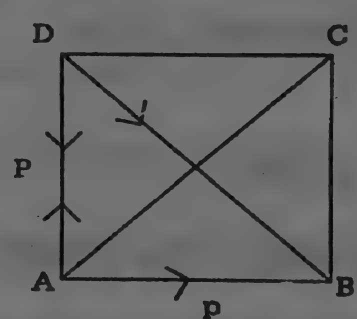
<u>Solution:</u>

Let the forces P and Q acting at A be represented in magnitude and direction by the lines AB and AD.

Complete the parallelogram BAD.

Then $P + Q = \overline{AB} + \overline{AD} + \overline{AC}$ (|| gm law)

$\therefore \overline{AC}$ is the sum of two forces.



$P - Q = \overline{AB} - \overline{AD}$

$= \overline{AD} + \overline{DA}$

$= \overline{DA} + AB$

$= DB$ (by Δ^{le} law)

$\therefore \overline{DB}$ is the difference of two forces.

It is given that $\overline{AC}$ and $\overline{DB}$ are at right angle.

(ie) in parallelogram ABCD, the diagonals AC and BD cut at right angles.

∴ ABCD must be a rhombus.

∴ AB = AD

(ie) P = Q in magnitude.

Space for Hints

Example: 2

A and B are two fixed points on a horizontal line at a distance C apart. Two fine light strings AC and BC of lengths b and a respectively support a mass at C.

Show that the tensions of the strings are in the radio b $(a^2 + c^2 - b^2)$: a $(b^2 + c^2 - a^2)$.

Solution:

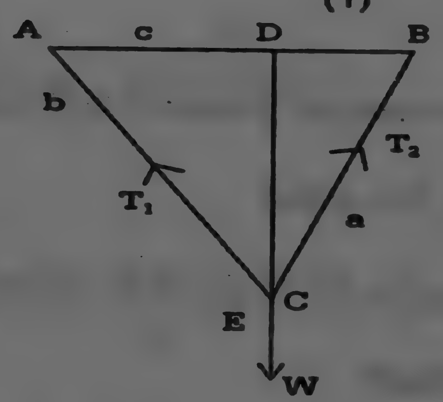
Let T_1 and T_2 be the tensions along the strings CA and CB and W the weight of the mass at C, acting vertically downwards along CE.

Produce EC to meet AB at D.

Since C is at rest under the action of three forces,

We have by Lami's theorem,

$$\frac{T_1}{\sin\angle ECB} = \frac{T_2}{\sin\angle ECA} \quad (1)$$



Now $\sin\angle ECB = \sin(180^0 - \angle DCB)$

$= \sin\angle DCB$

$= \sin(90^0 - \angle ABC)$

Space for Hints

$= \text{Cos } \angle ABC$

$\text{Sin } \angle ECA \quad = \text{Sin } (180^0 - \angle ACD)$

$= \text{Sin } \angle ACD$

$= \text{Sin } (90^0 - \angle BAC) = \text{Cos } \angle BAC$

Hence (1) becomes

$$\frac{T_1}{\cos \angle ABC} = \frac{T_2}{\cos \angle BAC}$$

$$\therefore \frac{T_1}{T_2} = \frac{\text{Cos}\angle ABC}{\text{Cos}\angle BAC} = \frac{\text{Cos}B}{\text{Cos}A} \qquad \textbf{(2)}$$

In ΔABC, we know that $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$ and

$$\text{Cos } A = \frac{b^2 + c^2 - a^2}{2bc}$$

Hence (2) becomes

$$\frac{T_1}{T_2} = \left(\frac{c^2 + a^2 - b^2}{2ca}\right) x \left(\frac{2bc}{b^2 + c^2 - a^2}\right)$$

$$= \frac{b(c^2 + a^2 - b^2)}{a(b^2 + c^2 - a^2)}$$

$\therefore$ The tensions of the strings are in the radio b $(a^2 + c^2 - b^2)$: a $(b^2 + c^2 - a^2)$.

Example:3

ABC is a given triangle. Forces P.Q,R acting along the lines OA, OB, OC are is equilibrium. Prove that

1)P:Q:R = $a^2 (b^2 + C^2 - a^2 - b^2)$: $b^2 (c^2 + a^2 - b^2)$: $c^2(a^2+b^2-c^2)$ if O is the circumcentre of the triangle.

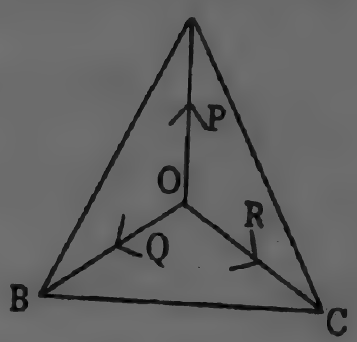
ii) P: Q: R = $\cos \dfrac{A}{2} : \text{Cos} \dfrac{B}{2} : \text{Cos} \dfrac{C}{2}$ if O is the in centre of the triangle.

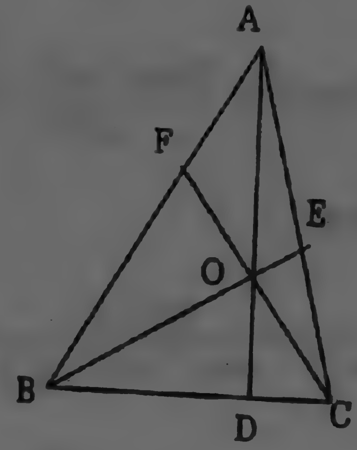
iii) P: Q: R = a: b: c if O is the ortho centre of the triangle.

iv) P: Q: R = OA: OB: OC if O is the centroid of the triangle.

Solution:

Space for Hints





By Lami's theorem,

We have

$$\frac{P}{\text{Sin}\angle BOC} = \frac{Q}{\text{Sin}\angle COA} = \frac{R}{\text{Sin}\angle AOB}$$

I) When O is the circum centre of the ΔABC.

$\angle BOC = 2\angle BAC = 2A$;

$\angle COA = 2B$

And $\angle AOB = 2C$

$\therefore$ (1) gives $\frac{P}{\text{Sin } 2A} = \frac{Q}{\text{Sin } 2B} = \frac{R}{\text{Sin } 2C}$.

(ie) $\frac{P}{2\,\text{Sin A Cos A}} = \frac{Q}{2\,\text{Sin B Cos B}} = \frac{R}{2\,\text{Sin C Cos C}}$

(2)

But $\cos A = \frac{b^2+c^2-a^2}{2bc}$ and $\text{Sin } A = \frac{2\Delta}{bc}$ where Δ is the area.

$\therefore 2\,\text{Sin A Cos A} = 2.\frac{2A}{b}.\frac{(b^2+c^2-a^2)}{2bc}$

$$= \frac{2\Delta(b^2+c^2-a^2)}{b^2c^2}$$

Space for Hints

III^{ly} $2\sin B\cos B = \dfrac{2\Delta(c^2+a^2-b^2)}{c^2a^2}$

and $2\sin C\cos C = \dfrac{2\Delta(a^2+b^2-c^2)}{a^2b^2}$

So (2) becomes

$$\frac{P.b^2.c^2}{2\Delta(b^2+c^2-a^2)}=\frac{Q.c^2.a^2}{2\Delta(c^2+a^2-b^2)}=\frac{R.a^2.b^2}{2\Delta(a^2+b^2-c^2)}$$

Multiplying throughout by $\dfrac{2\Delta}{a^2b^2c^2}$

We get

$$\frac{p}{a^2(b^2+c^2-a^2)}=\frac{Q}{b^2(c^2+a^2-b^2)}=\frac{R}{c^2(a^2+b^2-c^2)}$$

ii) When O is the in centre of the triangle, OB and OC are the bisectors of $\angle B$ and $\angle C$.

$$\therefore \angle BOC=180^0-\frac{B}{2}-\frac{C}{2}=180^0-\left(\frac{B}{2}+\frac{C}{2}\right)$$

$$\angle BOC = 180^0 - \left(90^0-\frac{A}{2}\right)=90^0+\frac{A}{2}$$

Similarly $\angle COA = 90^0 + \dfrac{C}{2}$

So (1) becomes.

$$\frac{P}{\operatorname{Sin}\left(90^0+\frac{A}{2}\right)}=\frac{Q}{\operatorname{Sin}\left(90^0+\frac{B}{2}\right)}=\frac{R}{\operatorname{Sin}\left(90^0+\frac{C}{2}\right)}$$

(ie) $\dfrac{P}{\operatorname{Cos}\frac{A}{2}}=\dfrac{Q}{\operatorname{Cos}\frac{B}{2}}=\dfrac{R}{\operatorname{Cos}\frac{C}{2}}$

iii) Let O be the orthocenter of the triangle in the fig.

In the fig.

AD, BE, CF are the altitudes.

Space for Hints

Quadrilateral AFOE in cyclic.

$$((\because \angle AFO + \angle AEO = 90^0 + 90^0 = 180^0)$$

$\therefore \angle FOE + A = 180^0$

(or) $\therefore \angle FOE = 180^0 - A$

III^{ly} $\angle COA = 180^0 - B$

And $\angle AOB = 180^0 - C$

Hence (1) becomes

$$\frac{P}{\text{Sin}(180^0 - A)} = \frac{Q}{\text{Sin}(180^0 - B)} = \frac{R}{\text{Sin}(180^0 - C)}$$

(ie) $\dfrac{P}{\text{Sin A}} = \dfrac{Q}{\text{Sin B}} = \dfrac{R}{\text{Sin C}}$

(ie) $\dfrac{p}{a} = \dfrac{Q}{b} = \dfrac{R}{c}\left(\text{Since}\dfrac{a}{\text{Sin A}} = \dfrac{b}{\text{Sin B}} = \dfrac{c}{\text{Sin C}}\right)$

iv) When O is the centriod of the triangle.

We know that

$$\Delta BOC = \Delta COA = \Delta AOB \text{ and each} = \frac{1}{3}\Delta ABC$$

$$\Delta BOC = \frac{1}{2} OB.OC \text{ Sin}\angle BOC = \frac{1}{3}\Delta ABC$$

$$\therefore \text{Sin}\angle BOC = \frac{2\Delta ABC}{3 OB.OC}; \text{Similarly} \text{Sin}\angle COA = \frac{2\Delta ABC}{3OC.OA}$$

$$\text{and Sin}\angle AOB = \frac{2\Delta ABC}{3OA.OB}$$

Hence (1) becomes

$$\frac{P.3OB.OC}{2\Delta ABC} = \frac{Q.3OC.OA}{2\Delta ABC} = \frac{R.3OA.OB}{2\Delta ABC}$$

(ie) $P.OB.OC = Q.OC.OA = R.OA.OB$

Dividing by OA. OB. OC throughout,

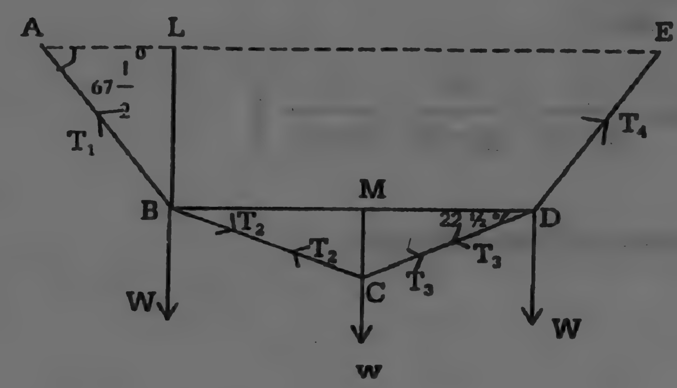
Space for Hints

$$\frac{P}{OA}=\frac{Q}{OB}=\frac{R}{OC}$$

Example: 4

Weights W, w, W are attached a points B, C, D respectively of a light string AE where B, C, D divide the string into 4 equal lengths. If the string hangs in the form of 4 consecutive sides of a regular octagon with the ends A and E attached to point on the same level, show that $W = \left(\sqrt{2+1}\right)w$.

Solution:



ABCDE is a part of regular octagon. We know that each interior angle of a regular polygon of n sides

$$= \left(\frac{2n-4}{n}\right) x\, 90^0$$

Putting n = 8, each interior angle of ABCDE

$$= \left(\frac{2 x 8-4}{8}\right) x\, 90^0$$

$$= \frac{12}{8} x 90^0 = 135^0$$

Let the tensions in the portions AB, BC, CD, DE be T_1, T_2, T_3, T_4 respectively. The string BC pulls B towards C and pulls C towards B, the tension being the same throughout its length. This fact is used to denote the

forces acting at B, C, and D.

In $\Delta BCD, \angle BCD = 135^0$

$\therefore \angle CBD = \angle CDB = \frac{45^0}{2} = 22\frac{1}{2}^0$

$\therefore \angle ABD = \angle ABC - \angle CBD.$

$= 135^0 - 22\frac{1}{2}^0 = 112\frac{1}{2}^0$

We know that every regular polygon is cyclic.

$\therefore$ A, B, C, D, E lie on the same circle.

$\angle EAB = 180^0 - \angle BDE$

$\therefore \angle EAB = 180^0 - \{\angle CDE - \angle BDC\}$

$= 180^0 - \{135^0 - 22\frac{1}{2}^0\}$

$= 45^0 + 22\frac{1}{2}^0 = 67\frac{1}{2}^0$

Equating the values of T_2 from (1) and (2)

We have

$$\frac{W}{\text{Sin}135^0}\text{Sin}\,22\frac{1}{2}^0 = \frac{w}{\text{Sin}135^0} = \text{Cos}\,22\frac{1}{2}^0$$

$$\text{(ie)}\frac{w}{W} = \tan 22\frac{1}{2}^0 = \sqrt{2-1}$$

$$\therefore W = \frac{w}{\sqrt{2-1}} = \frac{w(\sqrt{2+1})}{(\sqrt{2-1})(\sqrt{2+1})}$$

$$= \frac{w(\sqrt{2+1})}{1}$$

$$\therefore W = w\left(\sqrt{2+1}\right)$$

Space for Hints

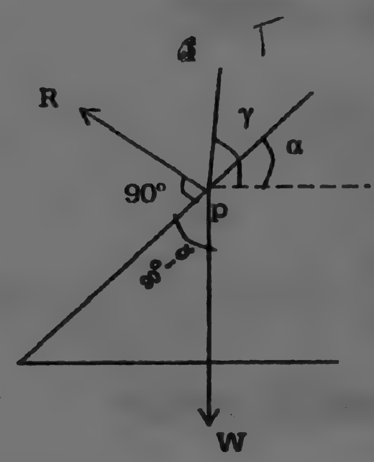
Example: 5

A weight is supported on a smooth plane of inclination α by a string inclined to the horizon at an angle γ. If the slope of the plane be increased to β and the slope of the string unaltered, the tension of the string is doubled. Prove that

$$\cot\alpha - 2\cos\beta = \tan\gamma.$$

Solution:

P is the position of the weight. The forces acting at P are i) its weight W downwards ii) the normal reaction R $\perp^r$ to the inclined plane and iii) the tension T along the string at an angle γ to the horizontal.



By Lami's theorem for the three forces at P,

$$\frac{T}{\operatorname{Sin}(180^0-\alpha)}=\frac{W}{\operatorname{Sub}(90^0-(\gamma-\alpha)}$$

$$\text{(ie)}\ \frac{T}{\operatorname{Sin}\alpha}=\frac{W}{\operatorname{Cos}(\gamma-a)}$$

$$\therefore T=\frac{W\operatorname{Sin}\alpha}{\operatorname{Cos}(\gamma-\alpha)} \qquad (1)$$

In the second case, the inclination of the plane is β.

There is no change in γ

If T_1 is the tension in the string.

$$\text{We have } T_1=\frac{W\operatorname{Sin}\beta}{\operatorname{Cos}(\gamma-B)}$$

Space for Hints

But $T_1 = 2T$ (given)

$$\therefore \frac{W \operatorname{Sin} \alpha}{\operatorname{Cos}(\gamma - \alpha)} = \frac{2W \operatorname{Sin}\beta}{\operatorname{Cos}(\gamma - \alpha)} \qquad [\because \text{ by (1) and (2)}]$$

$\therefore \operatorname{Sin}\beta \operatorname{Cos}(\gamma - \alpha) = 2 \operatorname{Sin}\alpha \operatorname{Cos}(\gamma - \beta).$

(ie) $\operatorname{Sin}\ \beta(\operatorname{Cos}\gamma \operatorname{Cos}\alpha + \operatorname{Sin}\gamma \operatorname{Sin}\alpha)$

$$= 2 \operatorname{Sin}\ \alpha(\operatorname{Cos}\gamma \operatorname{Cos}\beta + \operatorname{Sin}\gamma \operatorname{Cos}\beta)$$

$\operatorname{Sin}\ \beta \operatorname{Cos}\gamma \operatorname{Cos}\alpha = 2 \operatorname{Sin}\alpha \operatorname{Cos}\gamma \operatorname{Cos}\beta + \operatorname{Sin}\alpha \operatorname{Sin}\beta) \operatorname{Sin}\gamma$

$$= \operatorname{Sin}\ \alpha(2 \operatorname{Cos}\gamma \operatorname{Cos}\beta + \operatorname{Sin}\beta \operatorname{Sin}\gamma)$$

$$\therefore \frac{\operatorname{Cos}\alpha}{\operatorname{Sin}\alpha} = \frac{2 \operatorname{Cos}\gamma \operatorname{Cos}\beta + \operatorname{Sin}\beta \operatorname{Sin}\gamma}{\operatorname{Sin}\beta \operatorname{Cos}\gamma}$$

(ie) $\operatorname{Cot}\ \alpha = 2 \operatorname{Cot}\beta + \tan\gamma$ (or)

Cot There is no change in γ

If T_1 is the tension in the string.

$$\text{We have } T_1 = \frac{W \operatorname{Sin}\beta}{\operatorname{Cos}(\gamma - B)}$$

But $T_1 = 2T$ (given)

$$\therefore \frac{W \operatorname{Sin} \alpha}{\operatorname{Cos}(\gamma - \alpha)} = \frac{2W \operatorname{Sin}\beta}{\operatorname{Cos}(\gamma - \alpha)} \qquad [\because \text{ by (1) and (2)}]$$

$\therefore \operatorname{Sin}\beta \operatorname{Cos}(\gamma - \alpha) = 2 \operatorname{Sin}\alpha \operatorname{Cos}(\gamma - \beta).$

(ie) $\operatorname{Sin}\ \beta(\operatorname{Cos}\gamma \operatorname{Cos}\alpha + \operatorname{Sin}\gamma \operatorname{Sin}\alpha)$

$$= 2 \operatorname{Sin}\ \alpha(\operatorname{Cos}\gamma \operatorname{Cos}\beta + \operatorname{Sin}\gamma \operatorname{Cos}\beta)$$

$\operatorname{Sin}\ \beta \operatorname{Cos}\gamma \operatorname{Cos}\alpha = 2 \operatorname{Sin}\alpha \operatorname{Cos}\gamma \operatorname{Cos}\beta + \operatorname{Sin}\alpha \operatorname{Sin}\beta) \operatorname{Sin}\gamma$

$$= \operatorname{Sin}\ \alpha(2 \operatorname{Cos}\gamma \operatorname{Cos}\beta + \operatorname{Sin}\beta \operatorname{Sin}\gamma)$$

$$\therefore \frac{\operatorname{Cos}\alpha}{\operatorname{Sin}\alpha} = \frac{2 \operatorname{Cos}\gamma \operatorname{Cos}\beta + \operatorname{Sin}\beta \operatorname{Sin}\gamma}{\operatorname{Sin}\beta \operatorname{Cos}\gamma}$$

(ie) $\operatorname{Cot}\ \alpha = 2 \operatorname{Cot}\beta + \tan\gamma$ (or)

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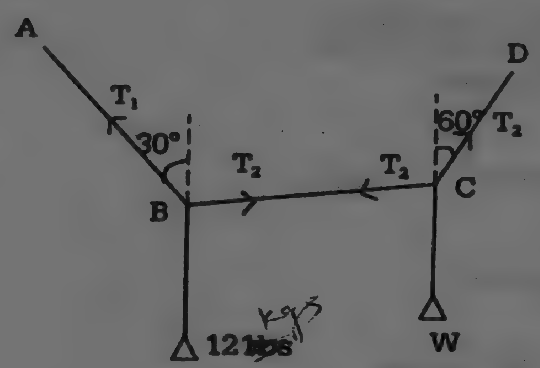
$\cot\alpha - 2\cot\beta = \tan\gamma$.

Example: 6

A string ABCD hangs from fixed points, A, D carrying a weight of 12kgs at B and A weight W at C. AB is inclined at 60^0 to the horizontal BC and CD is inclined at 30^0 to the horizontal. Find W.

Solution:

Let T_1, T_2, T_3 be the tensions in the Parts AB, BC, CD of the string ABCD hanging from the fixed points A and D.



Consider the equilibrium of the point B.

By Lami's theorem

$$\frac{T_1}{\sin 90^0} = \frac{T_2}{\sin(180^\circ - 30^0)}$$

$$= \frac{12}{\sin(90^0 + 30^0)}$$

$$\text{or } T_2 = 12\,\frac{\sin 30^0}{\cos 30^0} = \frac{12}{\sqrt{3}}$$

Consider next the equilibrium of C.

By Lami's theorem

$$\text{We get } \frac{T_2}{\sin(180^0 - 60)} = \frac{W}{\sin(90^0 + 60^0)} = \frac{T_3}{\sin 90^0}$$

or $W = T_2 \dfrac{\text{Cos}60^0}{\text{Sin}60^0} = \dfrac{12}{\sqrt{3}}.\dfrac{1}{\sqrt{3}} = 4\text{kgs.}$

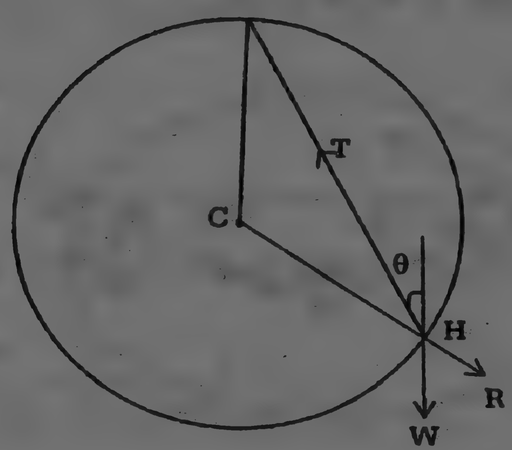
Space for Hints

Example: 7

A bead of weight w can slide on a smooth circular wire in a vertical plane the bead is attached by a light thread to the highest point of the wire and in equilibrium the thread is taut and makes an angle θ with the vertical. Find the tension of the thread and reaction of the wire on the bead.

Solution:

The bead is kept in equilibrium at a point A of the wire by three forces, $\overrightarrow{W}$ (the weight of the bead) the reaction $\overrightarrow{R}$ at A and the tension $\overrightarrow{T}$ in the string AB, where B in the highest point of the wire.



Also the wire being smooth the reaction $\overrightarrow{R}$ at A passes through C, the centre of the circular wire.

Since AB is inclined at an angle θ with the vertical, $\angle CBA = \theta$.

But BC = CA, So that $\angle CBA = \theta$

By Lami's theorem

$$\frac{W}{\text{Sin}(180^0 - \theta)} = \frac{R}{\text{Sin}(180^0 - \theta)} = \frac{T}{\text{Sin}2\theta}$$

or $\dfrac{W}{\text{Sin}\theta} = \dfrac{R}{\text{Sin}\theta} = \dfrac{T}{2\text{Sin}\theta\text{Cos}\theta}$

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This gives $T = 2W \cos\theta$ & $R = W$.

Example: 8

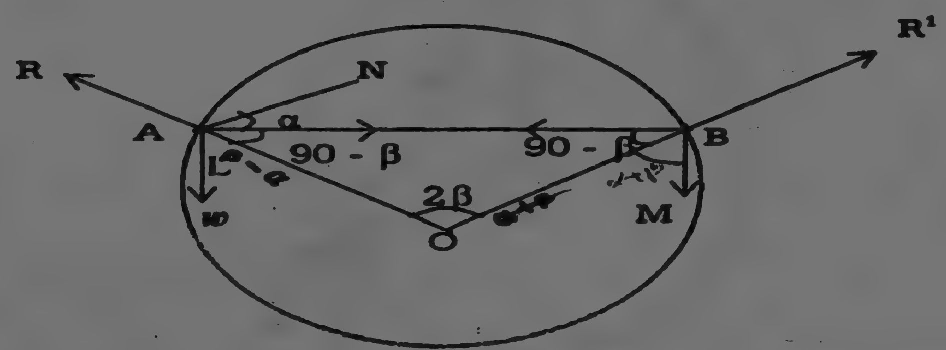
Two beads of weights w and w' can slide on a smooth circular wire in a vertical plane. They are connected by a light string which subtends an angle 2β at the centre of the circle when the beads are in equilibrium on the upper half of the wire. Prove that the inclination of the string to the horizontal is given by

$$\tan\alpha = \frac{W \sim W'}{w + w'}\tan\beta$$

Solution:

Let A and B be the beads of weights Wand W^1 connected by a light string and sliding on a circular wire.

In equilibrium position $\angle AOB = 2\beta$



O being the centre of the circle.

$\therefore \angle OAB = \angle OBA = 90^0 - \beta$.

Let AB make an angle α with the horizontal AN.

AL and BM are the vertical lines through A and B

$\angle ABM + \angle BAL = 180^0$

$\therefore \angle ABM = 180^0 - \angle BAL = 180^0 - (90^0 - \alpha) = 90^0 + \alpha$

$\therefore \angle OBM = \angle ABM - \angle ABO$

$\Rightarrow 90^0 + \alpha - (90^0 - \beta) = \alpha + \beta$.

The forces acting on the bead w at A are

i) Weight w acting vertically downwards along AL

ii) normal reaction R due to contact with the wire along the radius OA outwards. and

iii) tension T in the string along AB.

Similarly the force acting on the bead w' at B are

(i) weight w^1 acting vertically downwards along BM.

(ii) normal reaction R' along the radius OB outwards and

(iii) Tension T in the string along BA.

Applying Lami's theorem for the three forces at A

$$\frac{w}{\sin[180^0-(90-\beta)]}=\frac{T}{\sin(180^0-\beta-\alpha)}$$

(ie) $$\frac{w}{\cos\beta}=\frac{T}{\sin(\beta-\alpha)} \qquad (1)$$

Similarly applying Lami's theorem for the three forces at B.

$$\frac{w'}{\sin[180^0-(90-\beta)]}=\frac{T}{\sin(180^0-\overline{\beta+\alpha})}$$

(ie) $$\frac{w'}{\cos\beta}=\frac{T}{\sin(\beta+\alpha)} \qquad (2)$$

Dividing (1) by (2)

We have

$$\frac{w}{w'}=\frac{\sin(\beta+\alpha)}{\sin(\beta-\alpha)}$$

$$\frac{w-w'}{w+w'}=\frac{\sin(\beta+\alpha)-\sin(\beta-\alpha)}{\sin(\beta+\alpha)+\sin(\beta-\alpha)}$$

$$=\frac{2\cos\beta\sin\alpha}{2\sin\beta\cos\alpha}=\frac{\tan\alpha}{\tan\beta}$$

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$$\therefore \tan\alpha = \left(\frac{w-w'}{w+w'}\right)\tan\beta$$

Hence the result.

Exercises:

1. Three forces X, Y, Z, acting at the vertices A, B, C respectively of a triangle, each $\perp$ to the opposite side keep it in equilibrium. Prove that $\frac{X}{a}=\frac{Y}{b}=\frac{Z}{c}$.

1. If three forces represented in magnitude and direction by the bisectors of the angles of a Δ^{le} all acting from the vertices be in equilibrium, show that the triangle must be equilateral.

UNIT – 2

RESOLUTION OF A FORCE

Space for Hints

2.1 Introduction:

Two forces given in magnitude and direction have only one resultant for we can construct only one parallelogram when two adjacent sides are given. Conversely, a single force can be resolved into two components in an infinite number of ways. Since any number of parallelograms can be constructed on a given line AC as diagonal. (see fig 1. Section 1.2)

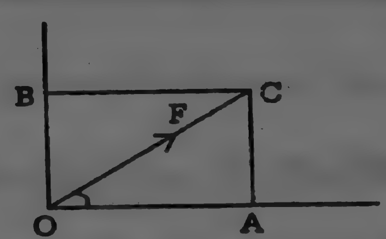
If BADC is any one of these, the force AC is equivalent to the two component forces AB and AD.

The most important case of resolution of a force occurs, when a given force is to be resolved in two directions at right angles, one of these directions being given. In this case, the magnitudes of the component forces are easily got as follows:

Let OC represent the given force F and OX be a line inclined at an angle Q to OC.

Let OY be perpendicular to OX.

Draw CA $\perp$ to OX and complete the parallelogram OACB.



(a)

Then the force OC is equivalent to the two component forces OA and OB.

Also OA = OC. Cos θ = F Cos θ

And OB = AC. = OC. Sin θ = F sin θ

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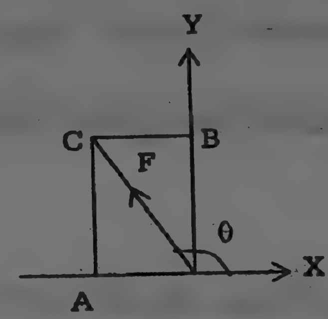
When a given force is resolved into two components two mutually perpendicular directions.

In fig. (a) OA is the resolved part of F along OX while OB is the resolved part of F along OY.

In fig. (b), θ is obtuse and OA is in a direction opposite to OX.

In this case, the resolved part of F along OX is negative.



(b)

Its value as before is F Cos θ Which is negative, as θ is obtuse.

Hence we have the following important proposition:

A force F is equivalent to a force F cos θ along a line making an angle θ with its own direction together with a force F sin θ perpendicular the direction of the first component.

Corollary: 1

When $\theta = O$, cos $\theta = 1$.

The resolved part = F

(ie) The resolved part of a force in its own direction is the force itself.

Corollary: 2

When $\theta = 90^0$, Cos $90^0 = 0$

The resolved part = O.

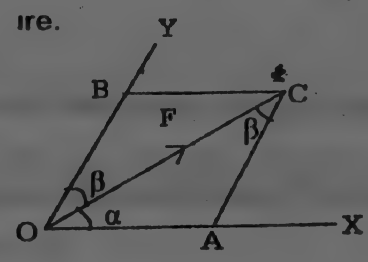
(ie) A force has no resolved part in a direction perpendicular to itself.

2.1.1 Components of a force along two given direction:

Let OC represent a given force F and OX, OY be two lines making

angles α and β with OC.

Draw CA parallel to OY and CB parallel to OX, making the parallelogram OACB as shown in the figure.



Then OA and OB are the components of the forces OC along OX & OY respectively.

From ΔOAC,

$$\frac{OA}{\text{Sin}\angle OCA}=\frac{AC}{\text{Sin}\angle AOC}=\frac{OC}{\text{Sin}\angle OAC}.$$

(ie) $$\frac{OA}{\text{Sin}\beta}=\frac{AC}{\text{Sin}\alpha}=\frac{OC}{\text{Sin}\{180-\overline{\alpha+\beta}\}}$$

$$\frac{OA}{\text{Sin}\beta}=\frac{AC}{\text{Sin}\alpha}=\frac{F}{\text{Sin}(\alpha+\beta)}$$

$$\therefore OA=\frac{F\,\text{Sin}\beta}{\text{Sin}(\alpha+B)}$$

and $OB = AC = \dfrac{F\sin\beta}{\text{Sin}(\alpha+\beta)}$

It should be noted that the component of force in a given direction is different from the resolved part of the force in that direction. To find the component of a force in any direction. We must be given the direction of the other component also. On the other hand to find the resolved part. We need only the given direction, since the other direction must be at right angles to it.

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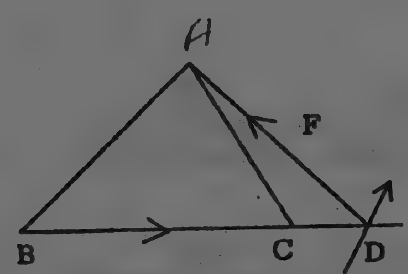
In other words the component of a force in a direction is a variable quantity, while the resolved part of the force in a direction is a fixed quantity its value being $F\cos\theta$.

Example: 1

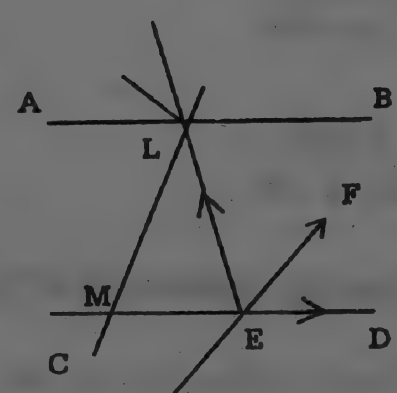
Show that a given force may be resolved into the components, acting in three given lines which are not all parallel or all concurrent.

Solution:

Let the three lines form a ΔABC and let the given force F meet the side BC in D.



Then F can be resolved into two components acting BC and DA respectively. The component along DA can be resolved into two components along AB and AC respectively.



Suppose two of the lines AB and CD are parallel and LM is the third line. Let the given force F meet CD at E. F can be resolved into two components along CD and EL. The component along EL can be resolved into two components acting along BA and ML respectively.

Example: 2

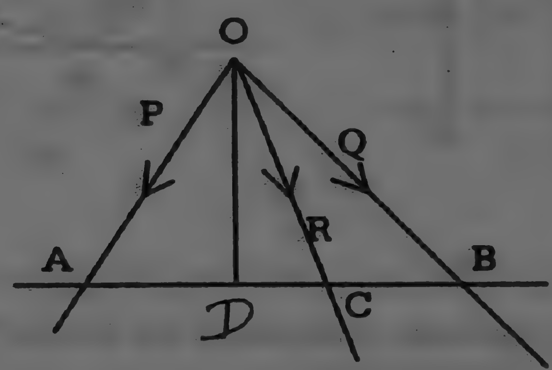
A line cuts the lines of action of three concurrent forces $\vec{P}, \vec{Q}, \vec{R}$ in

A,B,C respectively. If R the resultant of $\vec{P} \& \vec{Q}$ Show that $\frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}$.

Solution:

Let OD be drawn $\perp^r$ to the line ACB (the line cutting the lines of action of the forces) meeting it at D.

The resolved parts of $\vec{P}, \vec{Q}, \vec{R}$ along OD P Cos $\angle$AOD, Q Cos$\angle$BOD and R Cos$\angle$COD.



Since $\overline{R}$ is the resultant of $\overline{P}$ and $\overline{Q}$

We get

$$R \cos \angle COD = P \cos \angle AOD + Q \cos \angle BOD$$

Or R, $\frac{OD}{OC} = P\frac{OD}{OA} + Q.\frac{OD}{OB}$

(ie) $\frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}$

Exercises:

1. If a force P be resolved into two forces making angles of 45^0 and 15^0 with its direction show that the latter force is $\frac{\sqrt{6}}{3}$P.

2. Find the components of force P along two directions making angles of 45^0 and 60^0 with P on opposite sides.

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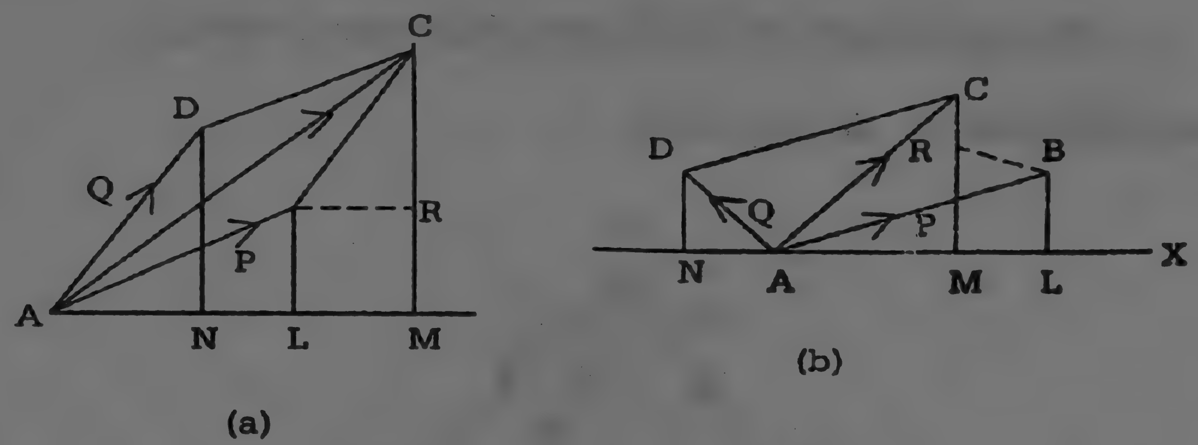
3. Two forces P and Q have a resultant R and the resolved part of R in the direction of P is of magnitude Q.

Show that the angle between the forces in $2\,\text{Sin}^{-1}\sqrt{\dfrac{P}{2Q}}$.

<u>2.2</u> <u>THEOREM ON RESOLVED PARTS</u>

The algebraic sum of the resolved parts of two forces in any direction is equal to the resolved part of the resultant in the same direction



Let AB and AD represent completely the forces P and Q and AX be the direction in which the forces are to be resolved. Complete the parallelogram ABCD so that the resultant R is represented by AC.

Draw BL, DN and CM perpendiculars to OX and BK $\perp$ to CM.

Then AL, AN and AM are resolved parts of the forces P,Q R along AX.

In fi. (b) AD makes an obtuse angle with AX and so the resolved part of Q is – AN.

We have to show that $AL \pm AN = AM$.

The triangles DAN and CBK are congruent and hence AN = BK

$\therefore AL \pm AN = AL \pm BK$

$= AL \pm LM = AM.$

Obviously the above theorem can be extended to the resultant of any number of forces acting at a point.

Suppose P_1, P_2, P_3 are three forces acting at O.

Let R, be the resultant of P_1 and P_2 and R_2 be the resultant of R_1 and P_3.

Applying the theorem to the two sets of three forces.

P_1, P_2, R_1 and R_1, P_3, R_2

We have

Resolved part of R_1 along OX = resolved part of P_1

+ Resolved part of P_2 (1)

And

Resolved part of R_2 along OX = Resolved Part of R_1

+ Resolved part of p_3 (2)

Combining (1) and (2)

We have

Resolved part of R_2 = resolved part of P_1 + resolved part of P_2

+ resolved part of $P_3$and so on.

Hence in a generalized form

We have the theorem

The algebraic sum of the resolved parts of a number of forces in any direction is equal to the resolved part of the resultant in the same direction.

In the application of this theorem, it is to be noted that all the forces are resolved in the same direction and each resolved part has to be taken with its proper sign.

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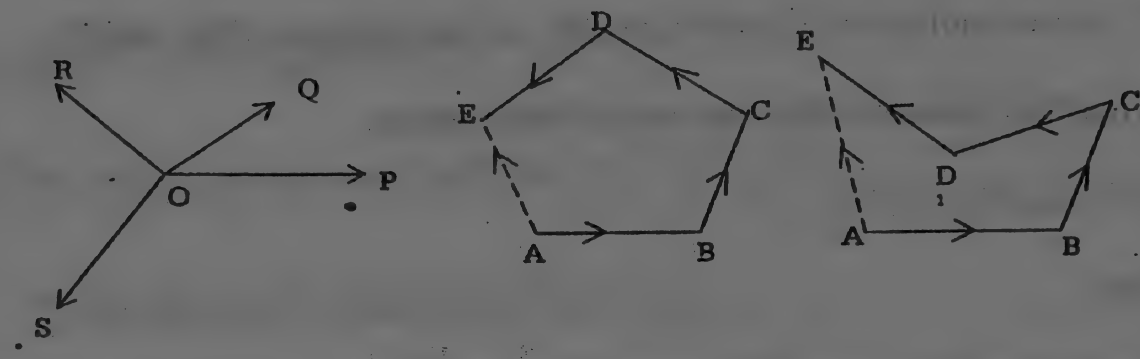
Resultant of any number of forces acting at a point: Graphical method:



Let P, Q, R, S be the forces acting at O.

Take a point A and draw lines AB, BC, CD and DE to represent successively the forces P, Q, R and S in magnitude and direction.

Compounding the forces by vector law, step by step we have

$$P + Q = \overline{AB}+\overline{AC}=\overline{AC}$$

$$P + Q + R = \overline{AC}+\overline{CD}=\overline{AD}$$

And $P + Q + R + S = \overline{AD}+\overline{DE}=\overline{AE}$

Hence the required resultant is represented in magnitude and direction by the line AE. The same construction will apply for any number of forces. The figure ABCDE is said to be the force – Polygon.

The force – polygon can be constructed by drawing the vectors corresponding to the forces in any order in fig (c) the order of the third and the fourth forces have been interchanged but AE is the same in each case.

Example: 1

ABC and $A'B'C'$ are two triangles. Show that the resultant of the forces $\overline{AA'}, \overline{BB'}$, and $\overline{CC'}$ acting on a particle is $3\overline{GG'}$ Where G and G' are the centroids of the two triangle.

Solution:

Since G is the centroid of ΔABC, by definition of $\overline{GA}+\overline{GB}+\overline{GC} = 0$

|||ly from the definition of G'. (1)

$$\overline{G'A'}+\overline{G'B'}+\overline{G'C'}=0. \quad (2)$$

But $\overline{AA'}=\overline{AG'}+\overline{G'A'}$

$$\overline{BB'}=\overline{BG'}+\overline{G'B'}$$

And $\overline{CC'}=\overline{CG'}+\overline{G'C'}$

Adding $\overline{AA'}+\overline{BB'}+\overline{CC'}=(\overline{AG'}+\overline{BG'}+\overline{CG'})+(\overline{G'A'}+\overline{G'B'}+\overline{G'C'})$

$$= \overline{AG'}+\overline{BG'}+\overline{CG'} \text{ by } (2)$$

$$=$$

$$(\overline{AG}+\overline{GG'})+(\overline{BG}+\overline{GG'})+(\overline{CG}+\overline{GG'})$$

$$= -(\overline{GA}+\overline{GB}+\overline{GC})+3\overline{GG'}$$

$$= 3\,\overline{GG'} \text{ by } (1)$$

Example: 2

Find a point inside a quadrilateral such that if it is acted on by forces represented by the lines joining it to the vertices of the quadrilateral. It will be in equilibrium.

Solution:

Let P be a point within the quadrilateral ABCD such that the forces represented by $\overline{PA},\overline{PB},\overline{PC},\overline{PD}$ are in equilibrium.

Let E and F be the mid points of the diagonals AC and BD of the quadrilateral.

Then $\overline{PA}+\overline{PB}+\overline{PC}+\overline{PD}$

$$= \left(\overline{PA}+\overline{PC}\right)+\left(\overline{PB}+\overline{PD}\right)$$

$$= 2.\ \overrightarrow{PE}+2.\overrightarrow{PF}=2\left(\overrightarrow{PF}+\overrightarrow{PF}\right)$$

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$= 2.2\ \overrightarrow{PM}$

Where M is the mid point of EF.

Since the forces are in equilibrium.

$$\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}+\overrightarrow{PD}=0$$

(or) $4\ \overrightarrow{PM}=0$

So that

$PM = 0$

(ie) P is coincides with M.

$\therefore$ the required point is the mid – point of the line joining the midpoints of the diagonals.

Example: 3

ABCD is a quadrilateral and forces acting at a point are represented in direction and magnitude by BA, BC, CD and DA. Find their resultant.

Solution:

We have $\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DA}=\overrightarrow{BA}$

$$\therefore \overrightarrow{BA}+\left(\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DA}\right)=\overrightarrow{BA}+\overrightarrow{BA}$$

$$= 2\ \overrightarrow{BA}$$

Hence the resultant is 2BA, both in magnitude and direction.

Exercises:

1. ABCDE is a pentagon. Forces acting on a particle are represented in magnitude and direction by AB, BC, CD, 2DE, AD and AE. Find their resultant.

2. a) ABCDE is a square and forces acting at a point are represented in magnitude and direction by AB, EBC, ECD and DA. Find their resultant.

b) Forces acting at a point are represented in magnitude and direction by $\overrightarrow{AB}, 2\overrightarrow{BC}, 2\overrightarrow{CD}, \overrightarrow{DA}$ and $\overrightarrow{DB}$ Where ABCD is a square. Show that the forces are in equilibrium.

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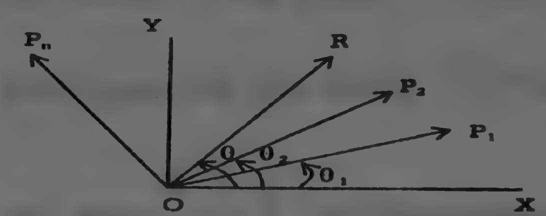
2.3 RESULTANT OF ANY NUMBER OF COPLANAR FORCES

Resultant of any number of coplanar forces acting at a point: Analytical method.

Let forces $P_1, P_2, P_3 \ldots P_n$ act at O. Through O draw two lines OX and OY at right angles to each other in the place of the forces.



Let the lines of action of $P_1, P_2 \ldots P_n$ makes angles $\alpha_1, \alpha_2 \ldots \alpha_n$ With OX.

Let R be the resultant inclined at an angle θ to OX.

Then R Cos θ = resolved part of the resultant along OX.

= algebraic sum of the resolved part of $P_1, P_2 \ldots Pn$ along OX.

[$\because$ by section 2:2]

$$= P_1 \cos\theta_1 + P_2 \cos + \ldots\ldots + P_n \cos\theta_n = X \text{ (say)} \qquad (1)$$

R sin θ = resolved part of the resultant along OY.

= algebraic sum of the resolved parts of $P_1, P_2 \ldots P_n$ along OY.

$$= Y \text{ (Say)} \qquad (2)$$

Squaring (1) and (2) and adding

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We have

$$R^2 = X^2 + Y^2$$

(ie) $R = \sqrt{x^2 + Y^2}$ (3)

Dividing (2) by (1)

$$\tan\theta = \frac{Y}{X}$$

(ie) $\theta = \tan^{-1}\left(\frac{Y}{X}\right)$ (4)

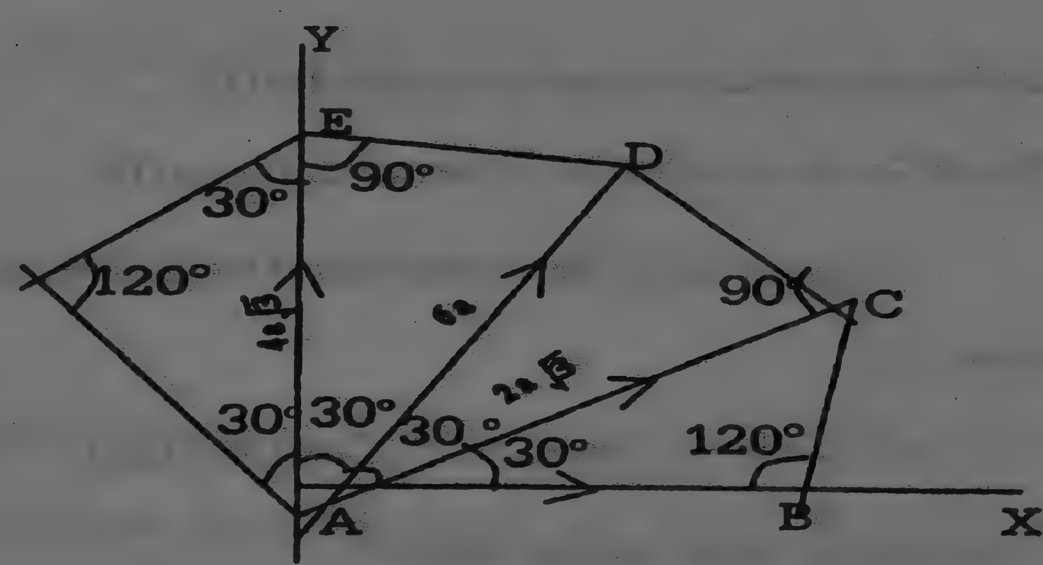
Equations (3) and (4) give respectively,

The magnitude and direction of the resultant.

Example: 1

ABCDEF is a regular hexagon and at A. act forces represented by $\overline{AB}, 2\overline{AC}, 3\overline{AD}, 4\overline{AE}$ and $5\overline{AF}$. Show that the magnitude of the resultant is AB. $\sqrt{351}$ and that it makes an angle $\tan^{-1}\left(\frac{7}{\sqrt{3}}\right)$ with AB.

Solution:



Let a be the side of the hexagon. Each interior angle of a regular hexagon = 120^0.

$\therefore \angle CAB = \angle ACB = 30^0$

$= \angle FAE = \angle FEA$

From the isosceles ΔABC

$AC = 2AB. \cos 30^0$

$$AC = 2a\,\frac{\sqrt{3}}{2}$$

$$= a\,\sqrt{3} = AE$$

$\angle AED = 90^0$

$\therefore\ AD^2 = AE^2 + ED^2$

$$= 3a + a^2 = 4a^2$$

$AD \quad = 2a.$

Since the vertices of a regular hexagon lie on a clicie.

$\angle DAB + \angle DCB = 180^0$

$\therefore \angle DAB = 180^0 - 120^0$

$$= 60^0$$

$\therefore \angle DAC = 30^0$ and $\angle EAD = 30^0$

The magnitudes of the forces acting at A are a, $2a\sqrt{3}$, 6a, $4a\sqrt{3}$ and 5a as shown in the figure.

Take AB and AE as axes of X and Y and let R be the resultant inclined at an angle θ to Resolving the forces along AB and AE.

We have

$$R\cos\,\theta = a + 2a\,\sqrt{3}\,\mathrm{Cos}30^0 + 6a\,\mathrm{Cos}\,60^0 + 5a\,\mathrm{Cos}\,120^0$$

$$= a + 2a\,\sqrt{3}.\,\frac{\sqrt{3}}{2} + 6a.\frac{1}{2} - 5a.\frac{1}{2}$$

$$= \frac{9a}{2} \qquad (1)$$

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and R Sin $\theta = 2a\sqrt{3}\,Cos60^0 + 6a\cos 30^0 + 4a\sqrt{3} + 5aCos30^0$

$$= 5a\sqrt{3+}\ \frac{11a\sqrt{3}}{2} = \frac{21a\sqrt{3}}{2} \qquad (2)$$

Squaring (1) and (2)

Add adding

$$R^2 = \frac{9a}{2} + \left(\frac{21a\sqrt{3}}{2}\right)^2$$

$$= \frac{81a^2}{4} + \frac{441}{4} \times 3a^2$$

$$= \frac{1404}{4} a^2 = 351a^2$$

$$R = a\sqrt{351}$$

Dividing (2) by (1)

$$\tan\theta = \frac{21a\sqrt{3}}{2} \times \frac{2}{9a}$$

$$= \frac{21\sqrt{3}}{9} = \frac{7\sqrt{3}}{3} = \frac{7}{\sqrt{3}}.$$

Hence the resultant is a force of magnitude AB $\sqrt{351}$, in a direction making an angle $\tan^{-1}\left(\frac{7}{\sqrt{3}}\right)$ with AB.

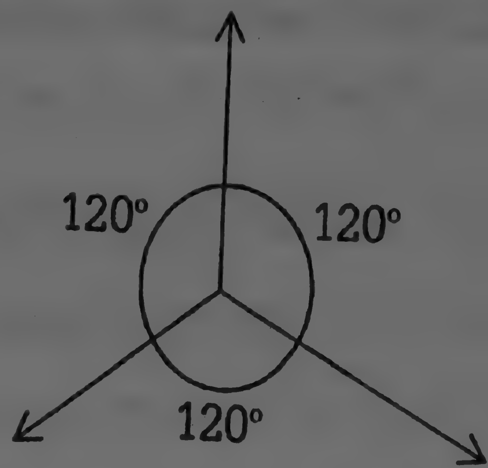
<u>Example: 2</u>

Find the magnitude and direction of the resultant of three coplanar forces P,2P, 3P acting at a point and inclined mutually at an angle of 120^0.

<u>Solution:</u>

Resolving the force along and perpendicular to the direction of the first force $\overline{P}$,

Space for Hints



We get

$$X = P \cos 0^0 + 2P \cos 120^0 + 3P \cos 240^0$$

$$= P - 2P \cos 60^0 - 3p \cos 60^0$$

$$= P - 2P\left(\frac{1}{2}\right) - 3P\left(\frac{1}{2}\right) = \frac{-3}{2}P.$$

$$Y = P \sin 0^o + 2P \cos 120^0 + 3P \cos 240^0$$

$$= P - 2P \cos 60^0 - 3P \cos 60^0$$

$$= P - 2P\left(\frac{1}{2}\right) - 3P\left(\frac{1}{2}\right) = \frac{-3}{2}P.$$

$$Y = P \sin 0^0 + 2P \sin 120^0 + 3P \sin 24^0$$

$$= 2P \sin 60^0 - 3P \sin 60^0 = -P \sin 60^0$$

$$= -P\frac{\sqrt{3}}{2}.$$

The resultant of the force has magnitude.

$$R = \sqrt{x^2 + Y^2}$$

$$= \sqrt{\frac{9}{4}P^2 + \frac{3}{4}P^2} = \sqrt{3}\,P$$

If θ is the angle made by this resultant with the first forces $\overline{P}$.

$$\text{Then } \tan\theta = \frac{Y}{X} = -\frac{\sqrt{3}}{2}P / -\frac{3}{2}P$$

Space for Hints

$= \frac{1}{3} = \tan 30^0 = \tan(180^0 + 30^0)$ Since X and Y are – ve, the resultant lies in 3^{rd} quadrant.

$\therefore \theta = 210^0$.

Example: 3

Forces of magnitudes P – Q, P + Q act at a point in directions parallel to the sides of an equilateral triangle taken in order. Show that the resultant is of magnitude Q $\sqrt{3}$ acting perpendicular to the direction of the second force.

Solution:

Let R be the resultant of force making an angle θ with BC. Resolving along BC.

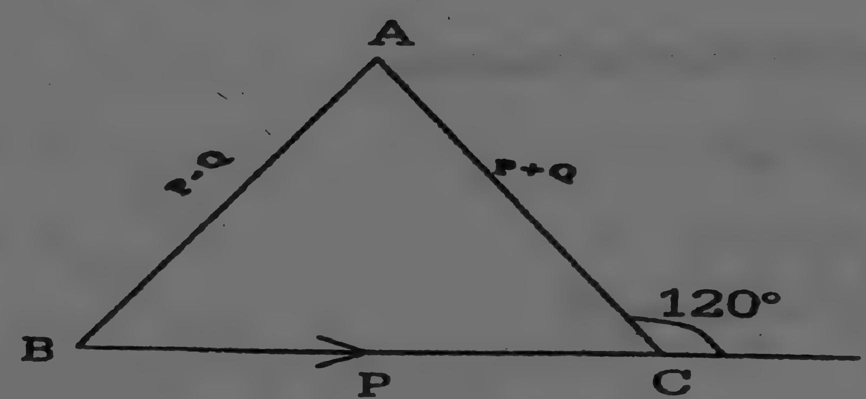
$$R \text{ Cos } = P \text{ Cos} 0^0 + (P+Q) \text{Cos} 120^0 - (P-Q) \text{ Cos } 60^0$$

$$= P + (P + Q) \text{ Cos } (180^0 - 60^0) - (P - Q) \text{ Cos } 60^0$$

$$= P + (P + Q) (-\text{Cos } 60^0) - (P - Q) \text{ Cos } 60^0.$$

$$= P - 2P \text{ Cos } 60^0$$

$$= P - P = 0 \qquad (1)$$



Resolving $\perp^r$ to BC

$$R \text{ Sin } \theta = P \text{ Sin } 0^0 + (P + Q) \text{ Sin } 120^0 - (P - Q) \text{ Sin } 60^0$$

$= (P + Q) \text{ Sin } (180^0 - 60^0) - (P - Q) \text{ Sin } 60^0$

$= (P + Q) \text{ Sin } 60^0 - (P - Q) \text{ Sin } 60^0.$

$= 2Q \text{ Sin } 60^0 = 2Q. \frac{\sqrt{3}}{2} = \sqrt{3.}\ Q.$

Squaring (i) & (ii) and adding

$R^2 (\text{Cos}^2 \theta + \text{Sin}^2\theta) = O + 3Q^2$

$R^2 = 3Q^2$

$R = \sqrt{3}\ Q$

Also, Dividing $\frac{R \sin\theta}{R \cos\theta} = \frac{\sqrt{3}Q}{O}$

$\text{Tan } \theta = \alpha$

$\theta = 90^0$

Resultant = $\sqrt{3}Q$ and acts $\perp r$ to BC ie $\perp r$ to the direction of P.

<u>Exercises:</u>

1. Forces P, 3P, $\sqrt{3}$ P and $\sqrt{3}$ P act along the straight lines OA, OB, OC, OD lying in a plane. If $\angle AOB = 60^0, \angle BOC = 90^0 \text{and} \angle COD = 120^0$, find the magnitude and direction of their resultant.

2. Forces of 2, $\sqrt{3}, 5\sqrt{3}$,2 kgs. Wt. Respectively act one of the angular points of a regular hexagon towards the five others in order. Find the direction and magnitude of the resultant.

3. Show that the resultant of forces equal to 7, 1, 1 and 3 kgs. wt. respectively acting at an angular point of a regular pentagon towards the other angular points, taken in order is $\sqrt{71}$ kgs. wt

Space for Hints

Space for Hints

4. Three forces P, Q, R in one plane act on a particle the angles between R and Q, P and R and P and Q being α, β and γ respectively show that their resultant is equal to $\sqrt{\{P^2 + Q^2 + R^2 + 2QR\cos\alpha + 2RP\cos\beta + 2PQ\cos\gamma\}}$

5. Three forces acting at a point are parallel to the sides of a triangle ABC, taken in order and proportional to the cosine of the opposite angles: Show that their resultant is proportional to $(1\text{-}8 \cos A \cos B \cos C)^{\frac{1}{2}}$.

2.4 <u>CONDITIONS OF EQUILIBRIUM</u>

Conditions of equilibrium of any number of forces acting upon a particle:

Forces acting at a point are in equilibrium when their resultant is zero. We shall now give the conditions which must be statisfied by a number of forces acting at a point of a rigid body or on a particle, in order that the body, or the particle may be at rest.

<u>Geometrical or Graphical Conditions:</u>

We have already studied the Triangle of Forces and the polygon of Forces.

If forces acting at a point are represented in magnitude and direction by lines forming the successive sides of a polygon, then for equilibrium, the polygon must be closed. When there are only three forces acting on a particle, the conditions of equilibrium are often most easily found by applying Lami's Theorem.

<u>Analytical Conditions:</u>

If we resolve the forces in any two directions at right angles and the sums of the components in these directions be X and Y, the resultant R is given by

$R^2 = X^2 + Y^2$ [Refer equation (3) is 2.3]

Space for Hints

If the forces are in equilibrium, $R = 0$.

Then $X^2 + y^2 = 0$

Now the sum of the squares of two real quantities cannot be zero unless each quantity is separately zero.

$\therefore\ X = 0$ and $Y = 0$.

Hence, if any number of forces acting at a point are in equilibrium, the algebraic sums of the resolved parts of the forces in any two perpendicular directions must be zero separately.

Conversely,

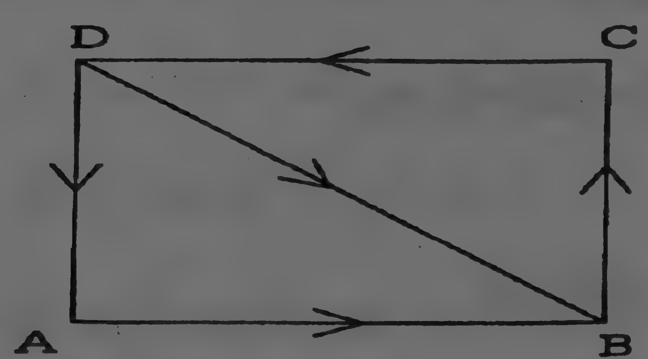
If the algebraic sum of the resolved parts of the forces acting at a point in any two perpendicular directions are zero separately the forces will be in equilibrium.

This is because when $X = 0$ and $Y = 0$ we must have $R = 0$.

Example: 1

Forces acting at a point are represented in magnitude and direction by $\overline{AB}, 2\overline{BC}, 2\overline{CD}, \overline{DA}$ and $\overline{DB}$ where ABCD is a square. Show that the forces are in equilibrium.

Solution:



$\overline{AB} + 2\overline{BC} + 2\overline{CD} + \overline{DA} + \overline{DB}$

$= (\ \overline{AB} + \overline{BC} + \overline{CD} + \overline{DA}\ ) + (\ \overline{BC} + \overline{CD} + \overline{DB}\ )$

Space for Hints

$= \overline{0}$ [Since the forces in the first set brackets are in equilibrium by the polygon of forces (square ABCD) and the forces in the second set are in equilibrium by the triangle of forces (triangle BCD)].

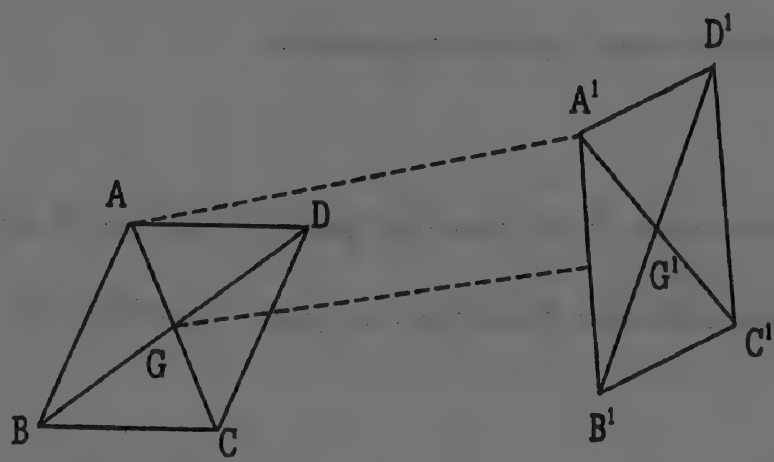
Hence the given set of forces are in equilibrium.

Example: 2

ABCD and $A'B'C'D'$ are Parallelograms prove that forces

$\overline{AA'}, \overline{B'B}, \overline{CC'}$ and $\overline{D'D}$ acting at a point will keep it at rest.

Solution:-



Let G and G^1 be the points of intersection of the diagonals.

By the polygon of forces, from the quadrilateral $AGG'A'$.

$AA' = \overline{AG} + \overline{GG'} + \overline{G'A'}$

Similarly

$\overline{B'B} = \overline{B'G'} + \overline{G'G} + \overline{GB}$

$\overline{C'C} = \overline{CG} + GG' + \overline{G'C'}$

$\overline{D'D} = \overline{D'G'} + \overline{G'G} + \overline{GD} (AG = AG')$

Adding up,

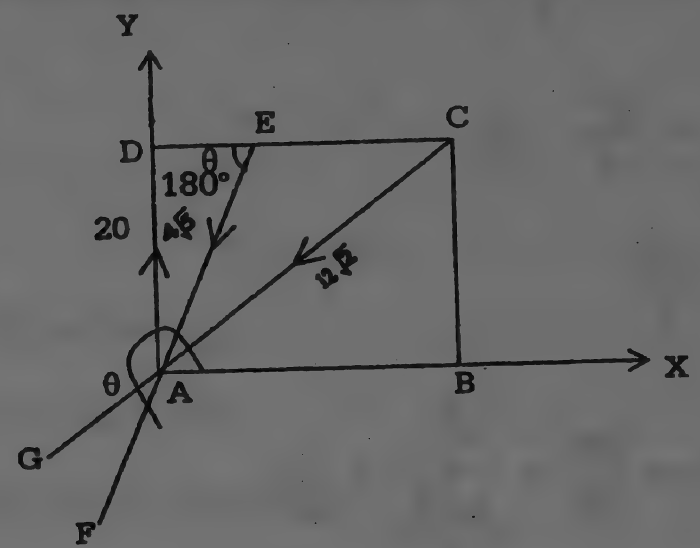
$\overline{AA'} + \overline{B'B} + \overline{CC'} + \overline{D'D} = \overline{O}$

[$\therefore$ the concerned vectors in the right side are equal and opposite].

Example: 3

E is the middle point of the side CD of a square ABCD. Forces 16, 20, $4\sqrt{5}, 12\sqrt{2}$ kg. wt. Act along AB, AD, EA, CA in the directions indicated by the order of the letters. Show that they are in equilibrium.

Solution:



Take AB and AD as axes of X and Y. Produce EA to F and let $\angle BAF = \theta$.

Produce CA to G.

$\angle BAG = \angle BAC + \angle CAG$

$= 45^0 + 180^0 = 225^0$

Let R be the resultant of the forces inclined at an angle θ to AB.

Resolving the forces along AB and AD.

We have

$R \text{ Cos } \theta = 16 + 12\sqrt{2} \text{ Cos } 225^0 + 4\sqrt{5} \text{ Cos } \theta.$

$= 16 + 12\sqrt{2} \text{ Cos } (180^0 + 45^0) + 4\sqrt{5} \text{ Cos } \theta$

$= 16 + 12\sqrt{2} \text{ x} - \text{Cos } 45^0 + 4\sqrt{5} \text{ Cos } \theta$

$= 16 - 12\sqrt{2} \text{ x } \sqrt{\frac{1}{2}} + 4\sqrt{5}\text{Cos}\theta$

$= 4 + 4\sqrt{5}\text{Cos}\theta$ (1)

Space for Hints

Space for Hints

$\angle BAE = \angle BAF - \angle EAF$

$= \theta - 180^0$

& $\angle DEA = \text{alt.}\angle BAE = \theta - 180^0$

In rt. θ d. ΔAED, $AE^2 = AD^2 + DE^2$

$$= a^2 + \frac{a^2}{4}, \text{ [a being the side of the square]}.$$

$$= \frac{5a^2}{4}$$

$$AE = \frac{a\sqrt{5}}{2}$$

$$\text{Cos}\,(\theta - 180^0) = \frac{DE}{AE} = \frac{(\frac{a}{2})}{(\frac{a\sqrt{5}}{2})} = \frac{1}{\sqrt{5}}$$

$$\text{Cos}\,(\theta - 180^0) = \frac{1}{\sqrt{5}} = -\text{Cos}\theta$$

$$\text{Cos}\,\theta = -\frac{1}{\sqrt{5}}$$

From (1)

$$R\,\text{Cos}\,\theta = 4 + 4\sqrt{5}\ x\ -\frac{1}{\sqrt{5}} = 0 \qquad (2)$$

$$R\,\text{Sin}\,\theta = 12\sqrt{2}\ \text{Sin}\,225^0 + 4\sqrt{5}\ \text{Sin}\,\theta + 20$$

$$= -\,12\sqrt{2}\ \text{Sin}\,(180^0 + 45^0) + 4\sqrt{5}\ \text{Sin}\,\theta + 20$$

$$= -\,12\sqrt{2}\ \text{Sin}\,45^0 + 4\sqrt{5}\ \text{Sin}\,\theta + 20$$

$$= -\,12\sqrt{2}\,.\,\frac{1}{\sqrt{2}} + 4\sqrt{5}\ \text{Sin}\,\theta + 20$$

$$= 8 + 4\sqrt{5}\ \text{Sin}\,\theta \qquad (3)$$

From rt $\angle$d ΔAED

Space for Hints

$$\mathrm{Sin}\,(\theta-180^0) = \frac{AE}{AE} = \frac{a}{\left(\frac{a\sqrt{5}}{2}\right)} = \frac{2}{\sqrt{5}}.$$

$$-\mathrm{Sin}\,(180^0-\theta) = \frac{2}{\sqrt{5}}$$

$$\text{(or)} \quad \mathrm{Sin}\,\theta = -\frac{2}{\sqrt{5}}$$

From (3) $R\,\mathrm{Sin}\,\theta = 8 + 4\sqrt{5} \times -\frac{2}{\sqrt{5}}$

$$= 8 - 8 = 0 \qquad (4)$$

Squaring (2) and (4) and adding.

$R^2 = 0 + 0 = 0$ (ie) $R = 0$

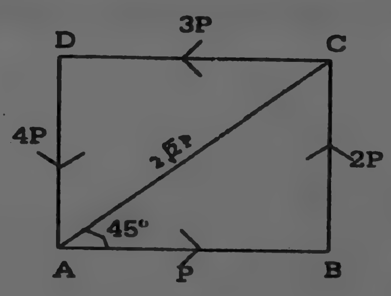
$\therefore$ The forces are in equilibrium.

Example: 4

Forces P, 2P, 3P, 4P and $2\sqrt{2}$ P act at a point in direction of AB, BC, CD, DA and AC where ABCD in square : Show that they are in equilibrium.

Solution:

Let R be the resultant making an angle θ with AB.



Resolving along AB

$R\,\mathrm{Cos}\,\theta = P - 3P + 2\sqrt{2}\,P\,\mathrm{Cos}\,45^0$

$$= -2P + 2\sqrt{2} + 2\sqrt{2}P.\frac{1}{\sqrt{2}} = 0$$

Space for Hints

Resolving $\perp^r$ to AB

$$R \sin \theta = 2P - 4P + 2\sqrt{2}P.\frac{1}{\sqrt{2}} = 0$$

Squaring and adding $R^2 = 0$

$$\therefore R = 0$$

Hence the forces are in equilibrium.

Exercises:

1. Fifteen Coplanar forces act at a point and are represented in magnitude and direction by the lines drawn from each of the vertices of a pentagon to the midpoints of those sides on which the vertex does not lie. Show that there are in equilibrium.
2. Three equal forces acting at a point are in equilibrium. Show that they are equally inclined to one another.
3. Three forces act perpendicularly to the sides of a triangle at their midpoints and are proportional to the sides. Prove that they are in equilibrium.

UNIT – 3

PARALLEL FORCES

3.1 Introduction

In the previous unit we have considered the method of finding the resultant of two forces which meet at a point. We shall now consider how to find the resultant of two parallel forces. Such forces do not meet in a point and so we cannot find their resultant by direct application of the law of parallelogram of forces.

Two parallel forces are said to be like when they act in the same direction. They are said to be unlike when they act in opposite parallel directions.

3.2 Resultant of two like and unlike parallel forces

To find the resultant of two like parallel forces acting on a rigid body:

Let like parallel forces P and Q act at the points A and B of the rigid body respectively and let them be represented by the lines AD and BL. At A and B, introduce two equal and opposite forces F of arbitrary magnitude along the line AB and let them be represented be AG and BN. These two new forces will balance each other and hence will not affect the resultant of the system.

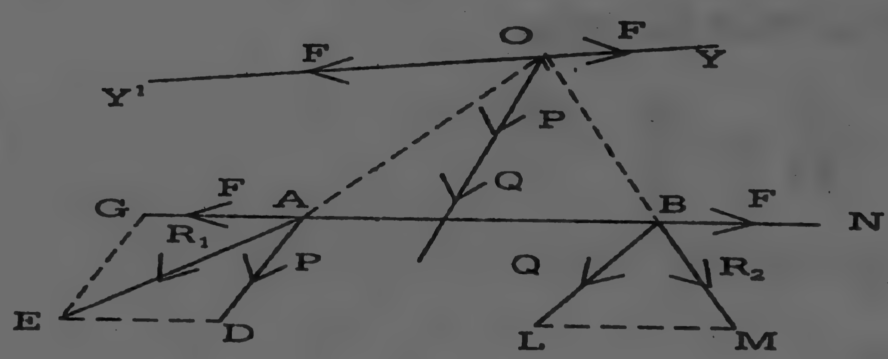


Fig. 1

Space for Hints

The two forces F and P acting at the point A can be compounded into a single force R, represented by the diagonal AE of the parallelogram ADEG. Similarly the two forces F and Q acting at the point B will have a resultant R_2 represented by the diagonal BM of the parallelogram BLMN.

Produce EA and MB and let them meet at O. The two resultants R_1 and R_2 can be considered to act at O. At O draw $Y'OY$ || to AB and OX || to the directions of P and Q Reresolve R_1 and R_2 at O into their original components.

R_1 at O is equal to a force F along OY' and a force P along OX. R_2 at O is equal to a force F along OY and a force Q along OX. The two Fs at O cancel each other, being equal and opposite. We are now left with two forces P and Q acting along OX. Hence their resultant is a force (P + Q) acting along OX.

(ie) acting in a direction parallel to the original directions of P and Q.

Thus the magnitude of the resultant of two like parallel forces is their sum. The direction of the resultant is parallel to the components and in the same sense.

To find the position of the resultant.

Let OX meet AB at C.

Triangles OAC and AED are similar.

$$\therefore \frac{OC}{AD} = \frac{AC}{ED}, \text{(ie)} \frac{OC}{P} = \frac{AC}{F}$$

$$\text{(or) } F.\ OC = P.\ AC \qquad (1)$$

Triangles OCB and BLM are similar.

$$\therefore \frac{OC}{BL} = \frac{CB}{LM}, \text{(ie)} \frac{OC}{Q} = \frac{CB}{F} \qquad (2)$$

$$\text{(or) } F.\ OC = Q.\ CB$$

From (1) and (2)

Space for Hints

We have P.AC = Q.CB

$$\text{(i.e.) } \frac{AC}{CB} = \frac{Q}{P}$$

(i.e.) The points C divides AB internally in the inverse ratio of the forces.

3.2.1 To find the resultant of two unlike and unequal parallel forces acting on a rigid body:

Let P and Q be two unequal and unlike parallel forces acting at the points A and B of the rigid body.

Let P > Q and let them be represented by AD and BL. At A and B introduce two equal and opposite forces F of arbitrary magnitude along the line AB and let them be represented by AG and BN. These two new forces will balance each other and hence will not affect the resultant of the system.

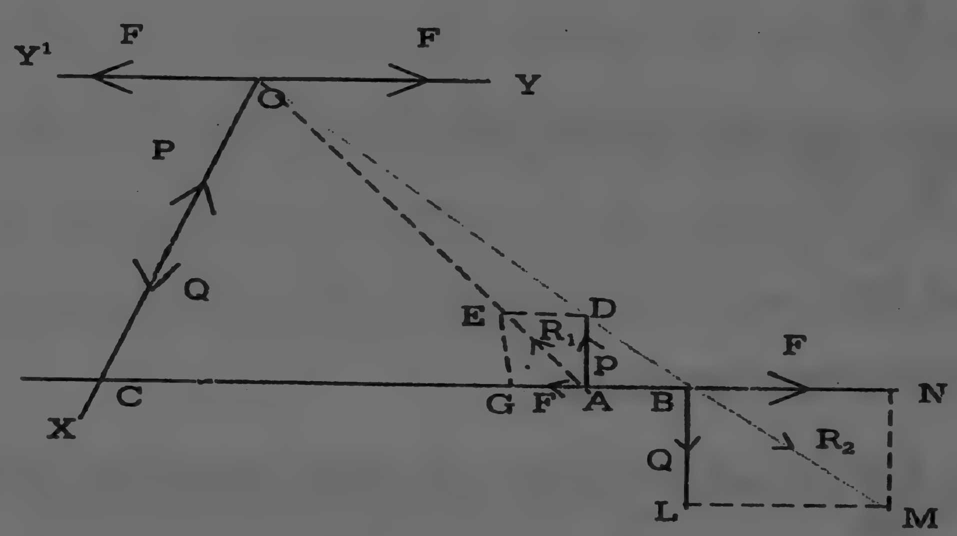


Fig. 2

The two forces F and P acting at A can be compounded into a single force R_1 represented by the diagonal AE of the parallelogram AGED. Similarly the two forces F and Q acting at B have a resultant R_2 by the diagonal BM of the parallelogram BLMN.

Produce AE and MB and let them meet at O. The two resultants R_1 and R_2 can be considered to act at O. At O, draw $Y'OY \parallel$ to AB and OX

Space for Hints

parallel to the directions of P and Q. Reresolve R_1 and R_2 at O into their original components. R_1 at O is equal to a force F long OY' and a force P along XO. R_2 at O is equal to a force F along OY and a force Q along OX. The two Fs at O cancel each other, being equal and opposite. We are not left with a force P along XO and a force Q along OX. Clearly the resultant is a force $P - Q$ (as $P > Q$) acting along XO. (ie) acting in a direction parallel to that of P.

Thus the magnitude of the resultant of two unlike parallel forces is their difference. The direction of the resultant is parallel to and in the sense of the greater component.

To find the position of the resultant:

Let OX meet AB at C.

Triangles OCA and EGA are similar

$$\frac{OC}{EG} = \frac{CA}{GA}$$

(ie) $\dfrac{OC}{P} = \dfrac{CA}{F}$

(or) F. OC = P. CA

Triangles OCB and BLM are similar.

$$\frac{OC}{BL} = \frac{CB}{LM}$$

(ie) $\dfrac{OC}{Q} = \dfrac{CB}{F}$

(or) F.OC = Q.CB

From (1) and (2)

We have P. CA = Q. CB

(ie) $\dfrac{CA}{CB} = \dfrac{Q}{P}$

(ie) The point C divides AB externally in the inverse ratio of the forces.

Space for Hints

Note: As P> Q CB must be > CA.

Hence the resultant passes nearer the greater force.

Failure of the above construction:

The construction for finding the resultant of two unlike parallel forces P and Q will fail.

If P = Q. (ie) if the forces are equal in magnitude. In that case, In. fig.2 triangles AGE and BNM will be congruent;

$\angle GAE = \angle NBM$ and the lines AE and MB will be parallel.

There will be no such point as O.

Hence we conclude that the effect of two equal and unlike parallel forces that the effect of two equal and unlike parallel forces cannot be replaced by a single force. Such a pair of forces have no single resultant and they constitute what is called a couple, which will be considered later on.

3.2.2 Resultant of a number of parallel forces acting on a rigid body:-

If a number of parallel forces P, Q, R.... act on a rigid body, their resultant can be found by successive applications of law of the parallelogram of Forces.

First, we find the resultant R_1, of P and Q then we find the resultant R_2 of R_1 and R this process is continued, until the final resultant is obtained. If the parallel forces are all like, the magnitude of the final resultant is the sum of the forces. If the parallel forces are not all like, the magnitude of the resultant is the algebraic sum of the forces each taken with its proper sign.

3.2.3 Condition of equilibrium of three coplanar parallel forces:-

Let P, Q, R be the three forces parallel in one place and be in equilibrium.

Space for Hints

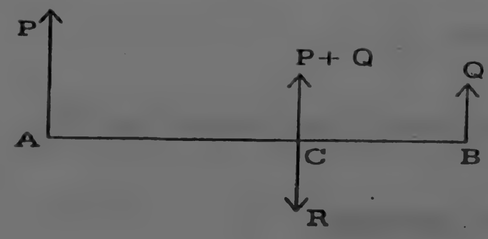


Fig. 3

Draw a line to meet the lines of action of these forces at A, B, and C respectively.

If all the three forces are in the same sense equilibrium will be like and the third R unlike.

The resultant of P and Q is (P+Q), Parallel to P or Q and hence, for equilibrium, R must be equal and opposite of (P+Q).

R = P+Q and the line of action of P+Q must pass through C.

P.AC = Q. CB

(ie) $\frac{P}{CB} = \frac{Q}{AC}$

and each $= \frac{P+Q}{CB+AC} = \frac{P+Q}{AB} = \frac{R}{AB}$

(ie) $\frac{P}{CB} = \frac{Q}{AC} = \frac{R}{AB}$

Thus, if three parallel forces are in equilibrium, each is proportional to the distance between the other two.

3.2.4: Center of two parallel forces:

Let P and Q be two parallel forces acting at two points A and B. Then, their resultant R passes through a point C. Which divides AB internally or externally in the ratio Q:P

(ie) $\frac{AC}{CB} = \frac{Q}{P}$

Space for Hints

The position of C given by (1) depends only upon the positions of A and B and then magnitudes of the forces P and Q.

It does not depend on the actual direction of P and Q. In other words, whatever be the common direction of parallelism of the forces P and Q their resultant will always pass through a certain fixed point. This fixed point is called the centre of the two parallel forces is a fixed point through which their resultant always passes whatever be the direction of parallelism.

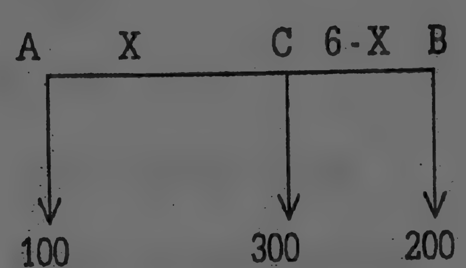
More generally, the resultant of a system of parallel forces of given magnitudes acting at given points of a body, will always pass through a fixed point, for all direction of parallelism. This point is called the centre of parallel forces.

Example: 1

Two men, one stronger, than the other have to remove a block of stone weighting 300 kgs. With a light pole whose length is 6 meter. The weaker man cannot

carry more than 100 kgs. Where must the stone be fastened to the pole. So as just to allow him his full share of weight?

Solution:



Let A be the weaker man bearing 100 kgs, his full share of the weight of the stone and B the stronger man bearing 200 kgs. Let C be the point on AB where the stone is fastened to the pole, such that AC = x. Then the

Space for Hints

weight of the stone acting at C is the resultant of the parallel forces 100 and 200 at A and B respectively.

$100\ AC = 200\ BC$

(ie) $100\ x = 200\ (6-x)$

$100\ x = 1200 - 200x$

$300\ x = 1200$

(or) $x = 4$

Hence the stone must be fastened to the pole at the point distant 4 metres from the weaker man.

Example: 2

Two like parallel forces P and Q act on a rigid body at A and B respectively.

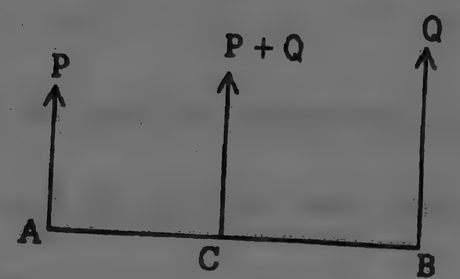
a) If Q be changed to $\frac{P^2}{Q}$, show that the line of action of the resultant is the same as it would be if the forces were simply interchanged.

b) If P and Q be interchanged in position, show that the point of application of the resultant will be displaced along AB through a distance d, where

$$d = \frac{P-Q}{P+Q} AB$$

Solution:

(a)



Let C be the centre of two parallel forces with P at A and Q at B.

Then P AC = Q. CB (1)

IF Q in changed to $\frac{P^2}{Q}$, (P remaining the same)

Let D be the new centre of parallel forces.

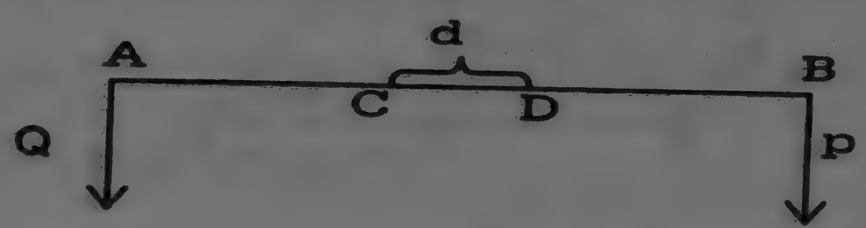
Then P. AD = $\frac{P^2}{Q}$.DB (2)

PQ.AD = P^2. DB

(or) Q. AD = P.DB (3)

Relation (3) shows that D is the centre of two like parallel forces with Q and A and P at B.

(b)



When the forces P and Q are interchanged in position, D is the new centre of parallel forces.

CD = d

From (3), Q. (AC + CD) = P. (CB - CD)

(ie) Q. AC + Q.d = P.CB – P.d

(or) (Q + P). d = P.CB – Q. AC

= P (AB – AC) – Q (AB - CB)

= P. AB – P. AC – Q.AB + Q.CB

(Q+P).d = (P-Q). AB (P. AC = Q.CB from (1))

(or) d = $\frac{P-Q}{P+Q.AB}$

Space for Hints

Space for Hints

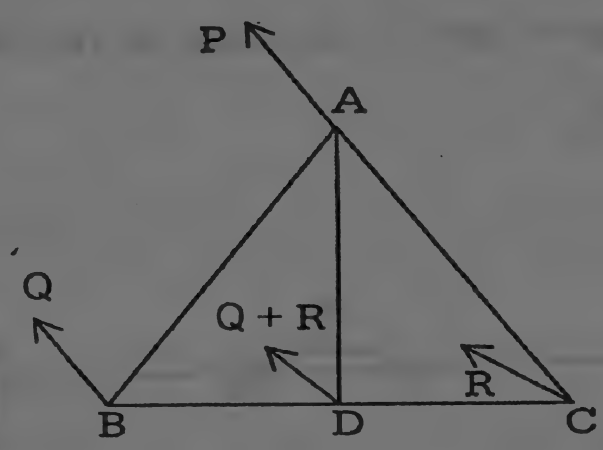
Example: 3

Three like parallel forces, acting at the vertices of a triangle, have magnitude proportional to the opposite to the opposite sides show that their resultant passes through the incentre of the triangle.

Solution:

Let like parallel forces P, Q, R act at A,B,C. It is given that

$$\frac{P}{a} = \frac{Q}{b} = \frac{R}{C} \qquad (1)$$



Let the resultant of Q and R meet BC at D.

We know that the magnitude of the resultant is Q + R.

(ie) Q. BD = R.DC

Also $\frac{BD}{DC} = \frac{\text{force at C}}{\text{force at B}} = \frac{R}{Q}$

$= \frac{c}{b}$ from (1)

$= \frac{AB}{AC}$

AD is the internal bisector of Δ A.

We have now to find the resultant of the two like parallel forces, Q+R at D and P at A.

Let this resultant meet AD at I.

Space for Hints

Then $\frac{AI}{ID} = \frac{\text{force at D}}{\text{force at A}} = \frac{Q+R}{P}$

$$= \frac{b+c}{a} \text{ from (1)}$$

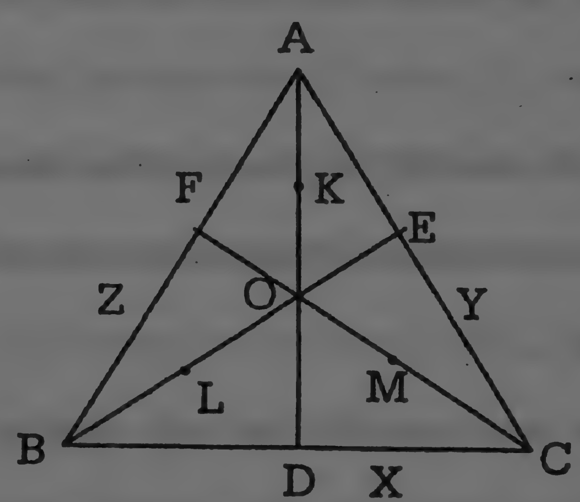
From result (2) if it is clear that I is the incetre of the Δ.

Example: 4

O is the orthocenter of a Δ ABC. Six equal like parallel forces pass, three of them through the middle points of the sides and the other three through the middle points of OA, OB, OC. Show that the resultant passes through the nine points centre of the triangle.

Solution:

Three forces each of magnitude P pass through the mid points X, Y, Z of the sides. Three other forces of magnitude P each pass through K, L, M the mid points of OA, OB and OC.



Space for Hints

The nine points circle passes through the points D, X and K and $\angle KDX = 90^0$. Hence KX is a diameter of the circle. The nine point is a diameter of the circle. The nine point center N is thus the middle point of KX. (ie) the resultant of like parallel forces $\vec{P}$ at k and $\vec{P}$ at X in $\overrightarrow{2P}$ acting at N. $|||^{ly}$ N being the midpoint of LY and MZ, the resultant of forces $\vec{P}$ each at L and Y as well as at M and Z will be $\overrightarrow{OP}$ and $\overrightarrow{2P}$ at N. (ie) the resultant of the six forces in $2\vec{P}+2\vec{P}+2\vec{P}=6\vec{P}$ at N the nine point centre.

Exercise:

1) If the magnitudes of two unlike parallel forces P,Q (P>Q) be increased by the same amount, show that the line of action of the resultant will move further off from P.
2) Three equal like parallel forces act at the middle points of the sides of a triangle, show that their resultant passes through the point of intersection.
3) Four equal like parallel forces act at the corners of a square, show that their resultant passes through the centre of square.

3.3 Moment of a Force:

When forces act on a particle, the only motion that can occur is a motion of translation. But a force acting on a rigid body may produce either a motion of translation or rotation of translation and rotation combined. When there is a motion of translation alone. The force is measured by the product of the mass of the particle and the acceleration produced on it by the force. In the case of rotation, the idea of the turning effect or moment of a force is introduced.

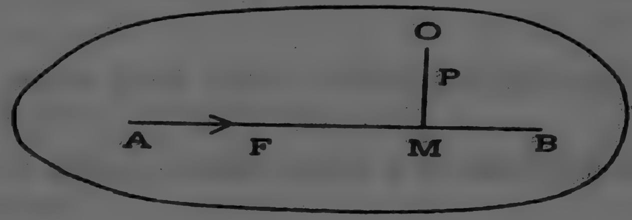


Fig. 1

Consider a sheet of cardboard pivoted freely at a fixed point O. if a force F acts along a straight line AB, it is clear that there will be no rotation if AB passes through O. If AB does not passes through O. The force will tent to rotate the sheet about O. This tendency to rotate the body will increase as the magnitude of the force increased and also as the perpendicular distance from O on the line of action of F. the tendency to rotate varies as F when ON is constant. It also varies ON when F is constant.

Hence it varies as FX. ON.

(ie) the product of F and ON. When both theses quantities vary.

This product is called the moment of F about O. Thus the moment of a force about a point is defined to be the product of the force and the perpendicular distance of the point from the line of action of the force.

The point F. ON will become Zero only if either F is Zero or ON is Zero.

When ON = O the point O is on the line of action of F.

Hence if the moment of a force about a point is zero, either.

i) the force itself is zero. Or

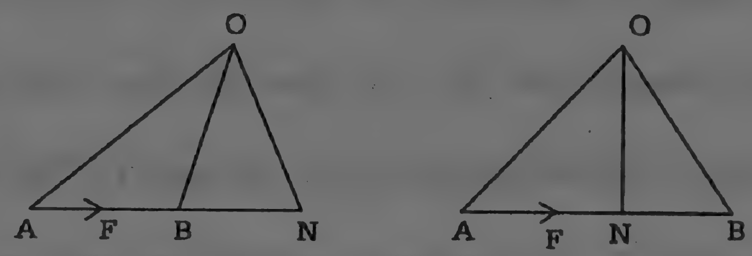
ii) the line of action of the force passes through that point.

3.3.1 Physical significance of the moment of a force:-

Space for Hints

From the definition of the moment of a force about a point, it is clear that it is a fitting measure for the turning effect of the force about that point. Thus the physical meaning for the moment of a force about a point is that is measures the tendency to rotate the body about the point.

3.3.2 Geometrical representation of a moment:-



(a) (b)

Fig. 2

Let a force F acting on a body be represented in magnitude direction and line of action by the line AB.

Let O be any given point and ON the perpendicular from O on AB or AB produced.

The moment of the force F about O.

$= F \times ON = AB \times ON = 2\ \Delta\ AOB.$

Hence if a force is represented completely by a straight line, its moment about any point is given by twice the area of the triangle which the straight line subtends at that point.

3.3.3 Sign of the moment:

In fig 2 (a) when the forces F acts along AB, it will tend to rotate the lamina in the anticlockwise direction.

(ie) in a direction opposite to that in which the hands of clock move. In such case, the moment is said to be positive. If the force tends to turn the body in a clockwise direction, its moment is said to be negative.

Thus the moment, of a force about a point has both magnitude and direction and in therefore a vector quantity.

3.3.4 Unit of moment:

The moment of a unit force about a point at a unit perpendicular distance from the line of action of force is defined as the unit for the measurement of moments. If the unit of force be a poundal and unit of distance be one foot of the unit of moment is a poundal foot. If the unit of force be a dyne and unit of distance be one centimeter, the unit of moment is a dyne – cm.

3.4 VARIGON'S THEOREM

Varigon's theorem of moments:

Statement:

The algebraic sum of the moments of two forces about any point in their plane is equal to the moment of their resultant about that point.

Proof:

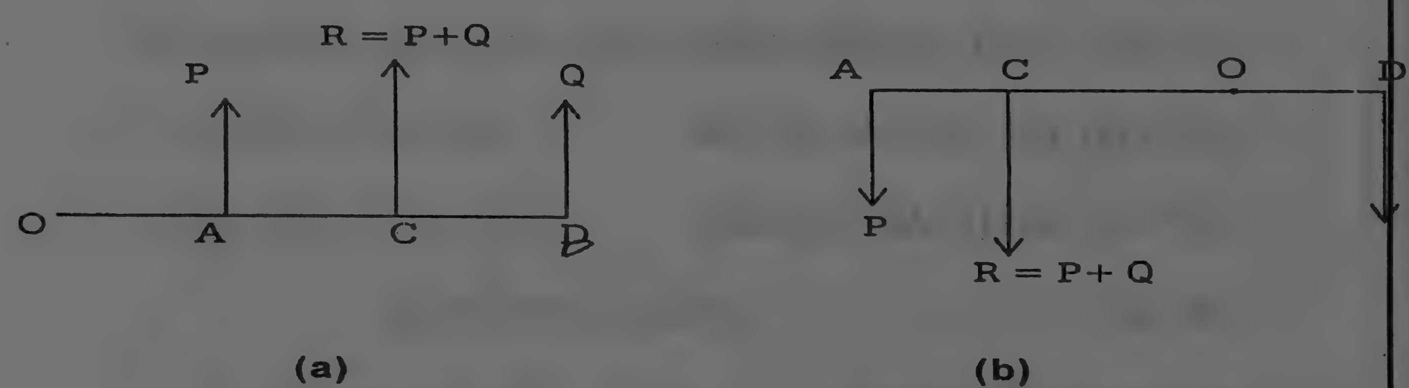


Fig. 3

To prove this theorem.

We consider two cases.

Case i) Let the forces be parallel.

Space for Hints

Let P and Q be two parallel forces and O any point in their plane. Draw AOB perpendicular to the forces to meet their lines of action in A and B.

The resultant of P and Q is a force

$R = (P + Q)$ acting at C. Such that

$P.AC = Q.\ CB.$

The algebraic sum of the moments of P and Q about O.

$= P.\ OA + Q.\ OB$

$= P.\ (OC - AC) + Q\ (OC + CB)$

$= (P+Q).\ OC - P.\ AC + Q.\ CB$

$= (P+Q).\ OC\ (P.AC = Q.\ CB)$

$= R.\ OC$

= moment of R about Q.

In fig (a) and (b). O is within AB and the algebraic sum of the moments of P and Q about O.

$= P.\ OA - Q.\ OB$

$= P.\ (OC + CA) - Q\ (CB - CO)$

$= (P + Q).\ OC + P.\ CA - Q.\ CB$

$= (P + Q).\ OC\ (P.\ AC = Q.\ CB)$

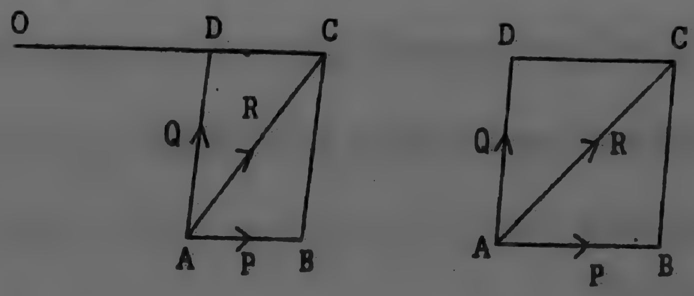
$= R.\ OC$

= moment of R about Q.

When the parallel forces P and Q are unlike and unequal, the theorem can be proved exactly in the same way.

Case ii)

Let the two forces P and Q act at A as shown in fig as (a) and (b)



(a) (b)

Fig. 4

And let O be any point in their plane. Through O, draw a line parallel to the direction of P meeting the line of action of Q at D.

Choose the scale of representation such that length AD represents Q in magnitude. On the same scale. Let length AB represents P. Complete the parallelogram BAD so that the diagonal Ac represents the resultant R of P and Q.

In either figure, the moments of P, Q, R about O are represented by $2\Delta AOB$, $2\Delta AOD$ and $2\Delta AOC$ respectively.

If fig (a), O lies outside the $\angle BAD$ and the moments of P and Q are both positive.

The algebraic sum of the moments of P and Q

$= 2\Delta AOB + 2\Delta AOD$

$= 2\ \Delta\ ACB + 2\ \Delta\ AOD$

$(\Delta\ AOB = \Delta\ ACB)$

$= 2\ \Delta\ ADC + 2\ \Delta\ AOD$

$(\therefore$ diagonal Ac bisects the $||_{gm})$

$= 2(\Delta\ ADC + \Delta\ AOD)$

$= 2\ \Delta\ AOC$

=moment of R about O.

In fig. (b) O lies inside the angle BAD.

Space for Hints

Space for Hints

The moment of P about O is positive while that of Q is negative.

The algebraic sum of the moments of P and Q

$$
\begin{aligned}
&= 2\Delta AOB - 2\Delta AOD \\
&= 2\Delta ACB - 2\Delta AOD \\
&= 2\Delta ADC - 2\Delta AOD \\
&= 2(\Delta ADC - \Delta AOD) \\
&= 2\Delta AOC \\
&= \text{moment of } R \text{ about } O.
\end{aligned}
$$

3.4.1 Generalised theorem of moments (principle of moments)

If any number of coplanar forces acting on a rigid body have a resultant the algebraic sum of their moments about any point in their plane is equal to the moment of their resultant about the same point.

Let P_1, P_2, P_3… etc., be any number of coplanar forces and O any point in their plane.

Let R_1 be the resultant of P_1 and P_2, R_2 that of R_1 and P_3. R_3 that of R_2 and P_4 and so on until the final resultant R is obtained.

Apply varigon's theorem to the forces P_1, P_2 and R_1

We have

Moment of P_1 about O + moment of P_2 about O = moment of R_1 about O… (1)

Similarly, applying the theorem to the forces R_1, P_3, and R_2.

We have

Moment of R_1 about O + moment of P_3 about O = moment of R_2 about O…(2)

Combining (1) and (2)

We have

Space for Hints

Moment of P_1 about O+ moment of P_2 about O + moment of P_3 about O = moment of R_2 about O.

Proceeding thus, till all the forces are exhausted, we prove the about theorem.

Let p_1, p_2, p_3... be the perpendicular distance of O from the lines of action of forces P_1, P_2, P_3 respectively and p. The perpendicular distance of O from the line of action of the resultant R.

$$P_1 p_1 + P_2, p_2, + \ldots = pR$$

$$\text{ie } \sum P_1 p_1 = pR \qquad (1)$$

From this theorem, we drive the following important corollaries:

Corollary: 1

If the basic point O about which moments is taken happens to lie on the line of action of the resultant R, then p=O.

$$\text{From (1) } \sum P_1 p_1 = O$$

Hence the algebraic sum of the moments of any number of coplanar forces about any point on the line of action of their resultant is zero.

Corollary 2:

$$\text{Suppose } \sum P_1 p_1 = 0$$

Then from (1) $pR = 0$

Either $p = o$ and $R = 0$

If $p = 0$, it means that the basic point O about which moment is taken, lies on the line of action of the resultant.

If $R = 0$, it means there is no resultant for system (ie) the forces are equilibrium.

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Thus if the algebraic sum of the moments of any number of forces about any point in the their plane is zero, then either their resultant passes through the point about which moments are taken or the resultant is zero. In the latter case, the forces will be in equilibrium.

Corollary: 3

Suppose R = 0

(ie) the forces are in equilibrium.

Then from (1) $\sum P_1 p_1 = p \times o = o$

Hence if a system of coplanar forces is in equilibrium the algebraic sum of their moments about any point in their plane is zero.

This theorem enables us to find points on the line of action of resultant of a system of forces. For, we have only to find a point which the algebraic sum of the moments of the forces is zero and then the resultant must pass through that point.

Example: 1

Two men carry a load of 224 kg. wt, which hangs from a light pole of length 8m. each end of which rests on a shoulder of one of the men. The point from which the load is hung is 2m. nearer to one man than the other. What is the pressure on each shoulder?

Solution:

AB is the light pole of length 8m. C is the point from which the load of 224 kgs is hung.

Let AC = x

Then BC = 8 – x

It is given that

Space for Hints

$(8 - x) - x = 2$

(ie) $8 - 2x = 2$

(or) $2x = 6$

$x = 3$

(ie) AC = 3 and BC = 5

Let the pressures at A and B be R_1 and R_2 kg. wt respectively.

Since the pole is in equilibrium, the algebraic sum of the moments of the three forces R_1, R_2, and 224 kg. wt. about any point must be equal to zero.

Taking moments about B,

$224\ CB - R_1\ AB = 0$

(as the moment of R_2 about B is o)

(ie) $224 \times 5 - R_1 . 8 = 0$

$$R_1 = \frac{224x5}{8}$$

$R_1 = 140$.

Taking moments about A,

$R_2\ AB - 224.\ AC = O$

(ie) $8R_2 - 224 \times 3 = o$

$$R_2 = \frac{224x3}{8}$$

$R_2 = 84$

<u>Note: 1</u>

For equilibrium, the weight of 224 kgs must be equal and opposite to the resultant of R_1 and R_2.

$R_1 + R_2 = 224$.

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Hence from this relation, we may find R_2 after finding R_1.

Note: 2

In practice, instead of equating the algebraic sum of the moments of the forces about any point to zero, it will be convenient to equate the sum of the moments in one direction to the sum of the moments in the other direction. Hence in the above, taking moments about B,

We have R_1. AB = 224. BC.

Example: 2

A uniform plank of length 2a and weight W is supported horizontally on two vertical props at a distance b apart. The greatest weight that can be placed at the two ends in succession without upsetting the plank are W_1 and W_2 respectively. Show that

$$\frac{W_1}{W+W_1}+\frac{W_2}{W+W_2}=\frac{b}{a}$$

Solution:

Let AB be the plank placed upon two vertical props at C and D. CD = b

The weight w of the plank acts at G, the midpoints of AB,

AG = GB = a

When the weight W_1 is placed at A, the contact with D is just broken and the upward reaction at D then is zero.

There is upward reaction R_1 at C.

Now, taking moments about C.

We have W_1. AC = W. CG

(ie) W_1 (AG - CG) = W. CG

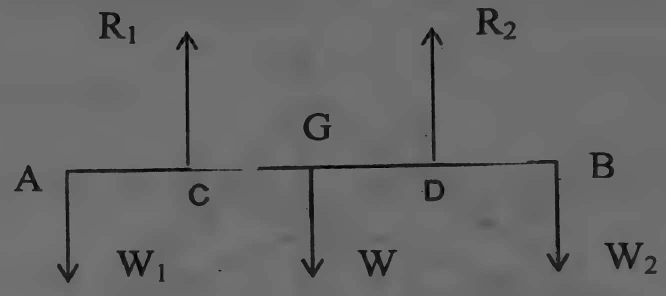
W_1 AG = $(W + W_1)$ CG

Space for Hints

(ie) $W_1 a = (W + W_1) CG$

$$CG = \frac{W_1 a}{W + W_1}$$

(1)



When the weight W_2 in attached at B,

There is loose contract at C. The reaction at C becomes zero. There is upward reaction R_2 about D.

Now taking moments about D.

We get $W.GD = W_2\ BD$

(ie) $W.\ GD = w_2\ (GB - GD)$

$GD\ (W + W_2) = W_2\ GB = W_2.a$

$$GD = \frac{W_2 a}{W + W_2} \qquad (2)$$

But $CG + GD = CD = b$

$$\frac{W_1 a}{W + W_1} + \frac{W_2 a}{W + W_2} = b$$

$$\frac{W_1}{W + W_1} + \frac{W_2}{W + W_2} = \frac{b}{a}$$

Example: 3

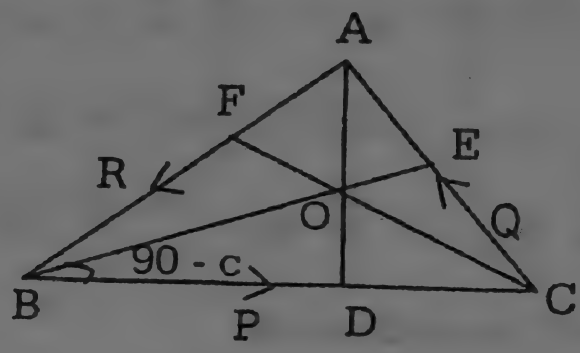
The resultant of three forces P, Q, R acting long the sides BC, CA, AB of a triangle ABC passes through the orthocentre. Show that the triangle must be obtuse angled. If $\angle A = 120^\circ$, and B = C, show that $Q+R = P\ \sqrt{3}$.

pace for Hints

Solution:

Let AD, BE and CF be the altitudes of the triangle intersection at O. the orthocentre.

As the resultant passes through O, moment of the resultant about O = 0.



$\therefore$ Sum of the moments about P, Q, R about O is also = O.

Hence, taking moments about O.

We have P. OD + Q. OE + R. OF = O. (1)

In rt. $\angle$d ΔBOD.

$$\angle OBD = \angle EBC = 90^0 - C$$

$$\therefore \tan(90^0 - c) = \frac{OD}{BD} \quad (2)$$

$$\therefore \text{Cot}C = \frac{OD}{BD}$$

Or OD = BD Cot C

From rt. $\angle$d ΔABD,

$$\text{Cos B} = \frac{BD}{AB}$$

$\therefore$ BD = AB.Cos B = c.Cos B.

$\therefore$ From (2), OD = c Cos B. Cot C

$$= c \text{ Cos B } \frac{\text{Cos}C}{\text{Sin}C}$$

$$= \frac{c}{\text{Sin}C}.\text{Cos B Cos C}$$

Space for Hints

$= 2R'\ \text{Cos B Cos C}\ (\because \frac{c}{\text{SinC}} = 2R', R'$ being the circumradius of the Δ)

Similarly OE = 2R′ Cos C Cos A.

And OF = 2R′ Cos A Cos B.

Hence (1) becomes

P. 2R′ Cos B cos C + Q. 2R′ Cos C A + R. 2R′ Cos A Cos B = O

Dividing by 2R′ Cos A Cos B Cos C

We get $\frac{P}{\text{CosA}} + \frac{Q}{\text{CosB}} + \frac{R}{\text{CosC}} = O$ (3)

Now P, Q, R being magnitudes of the forces, are all positive.

Hence in order that relation (3) may hold good, at least one of the terms must be negative.

Hence one of the cosines must be negative.

(ie) the triangle must be obtuse angled.

(Two of the cosine cannot be negative, as we cannot have two obtuse angles in the same triangle).

If $A = 120^0$ and the other angles equal,

then $B = C = 30^0$

Hence (3) becomes

$$\frac{P}{\text{Cos}120^0} + \frac{Q}{\text{Cos}30^0} + \frac{R}{\text{Cos}30^0} = O$$

(ie) $\frac{P}{(-\frac{1}{2})} + \frac{Q+R}{(\frac{\sqrt{3}}{2})} = O$

(ie) $P\sqrt{3} = Q + R$

Space for Hints

Example: 4

Forces P,Q,R act along the sides BC, AC BA respectively of an equilateral triangle. If their resultant is a force parallel to BC through the centroid of the triangle, prove that $Q = R = \frac{1}{2}P$.

Solution:

Δ ABC being equilateral the medians AA^1. BB^1 and CC^1 are also the altitudes meeting at G, the centroid.

Let DE be || to BC through G.

It is given that DGE is the line of action of the resultant.

As the resultant passes through G, its moment about G = 0

Sum of the moments of P,Q, R about G is also = 0.

(ie) $P.\ G\ A' - Q.\ GB' - R.\ GC' = 0$.

(ie) $P - Q - R = 0$ $\qquad (GA' = GB' = GC')$ $\quad$ (1)

Since the resultant passes through E also, sum of the moments of P, Q, R about E is = O.

Draw EL $\perp$ to BC and EM $\perp$ to AB.

$P.\ EL - R.\ EM = 0$ $\qquad$ (2)

From the similar Δs ELC and AA′C,

$$\frac{EL}{AA'} = \frac{EC}{AC} = \frac{1}{3}$$

$$(\therefore DGE \parallel BC \text{ and } \frac{AD}{DB} = \frac{AE}{EC} = \frac{AG}{GA'} = \frac{2}{1})$$

$$\therefore EC = \frac{1}{3}AA' \qquad (3)$$

From the similar Δs AME and AC′C.

$$\frac{EC}{CC'} = \frac{AE}{AC} = \frac{2}{3}$$

$$EM = \frac{2}{3}CC' \qquad (4)$$

$\therefore$ Substituting (3) and (4) in (2)

We have

$$P.\frac{1}{3}AA' - R.\frac{2}{3}CC' = 0$$

(ie) $P = 2R$ ($\because AA' = CC'$)

$$\text{(or) } R = \frac{P}{2} \qquad (5)$$

Putting $R = \frac{P}{2}$ in (1)

We have $P - Q - \frac{p}{2} = O$

(ie) $\frac{p}{2} - Q = O$

$$\frac{P}{2} = Q$$

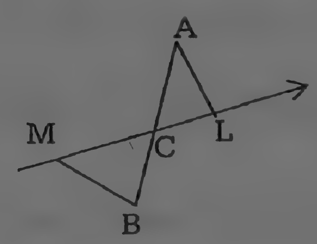
From (5) and (6) $R = \frac{P}{2} = Q$

<u>Example: 5</u>

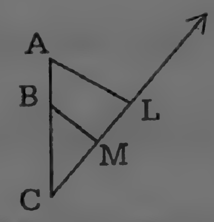
If ℓ and m are the moments of a given force P and two fixed points A and B respectively. Show that the line of action of divides AB in the ratio ℓ:M deduce that a force is completely known when the moments of the force about any three points not lying in the same straight line, are known.

Space for Hints

Solution:



(a)



(b)

Let ℓ and m be the moments of a given force P and about A and B and let them be in

i) Opposite directions, as shown in fig (a)

ii) The same direction, as shown in fig (b)

Draw AL and BM perpendicular to the line of action of P

Let the line of action of P meet AB or AB produced at C

ℓ = P.A ℓ and m = P.BM

$\frac{\ell}{m} = \frac{AL}{BM} = \frac{AC}{CB}$ (Δ s ACL and BCM are similar)

Hence the line of action of P divides AB in the ratio l:m internally in fig (a) and externally in fig (b).

Let ℓ, m, n be the moments of a force P about three given non-collinear points A, B, C respectively. The line of action of P divides AB internally or externally in the ration ℓ:m. Similarly it divides BC internally or externally in the ration m:n. Hence we can fix the points D and E in which the line of action of P intersects AB and BC respectively.

The line of action of P in therefore the straight line DE.

Let p be the perpendicular distance of A from DE.

Then P.p = ℓ.

$P = \frac{\ell}{\rho}$

As ℓ and p are known, we can calculate the magnitude of the force P.

Thus the force is known completely.

Example: 6

Find the locus of all points in a plane such that two forces through a point given in magnitude and position on the plane shall have equal moments about them in the same sense.

Solution:

Let $\overrightarrow{OA}, \overrightarrow{OB}$ be two forces of given magnitude and position. Join AB and let ℓ be a line through O, parallel to AB.

Let P be any point on ℓ.

Then the moments about P of $\overrightarrow{OA} \text{and} \overrightarrow{OB}$ are of the same sense and their magnitudes are equal to 2 Δ POA and 2 Δ POB.

But Δ POA = Δ POB, as the triangle stand on the same base OP between the same parallels.

Hence the moments about P are equal in magnitude and sign.

The locus of such points in hence the line ℓ.

3.4.2 Moment of a force about an axis:

We have considered only coplanar forces and their moments about a point in their plane. Let us now consider a rigid body which in capable of turning about some axis fixed in the body for instance a door capable of turning about the line of things. Now, any force whose line of action is not parallel to or does not passes through this axes, will tend to turn the body about it. To measure the tendency of rotation in such cases we introduce the idea of the moment of a force about an axis.

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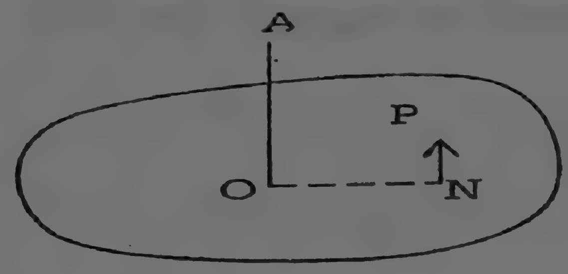
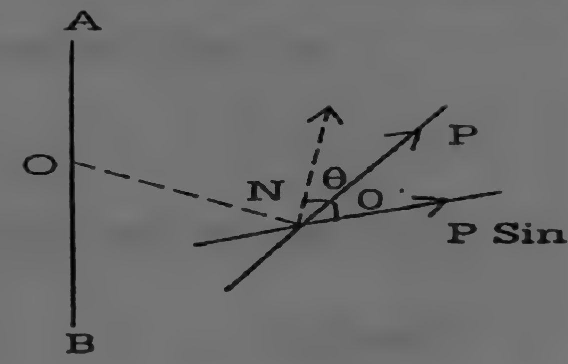


(a)



(b)

In fig. (a) a force P acts on a body in a direction perpendicular to a line AB in the body, but not intersecting it.

(ie) P acts in a plane $\perp^r$ to AB. The moment of the force P about the line AB is defined to be P. ON. Where ON is $\perp^r$ distance between the line of action of P, and the line AB.

In fig. (b) the force P acts in any direction (not necessarily) $\perp^r$ to AB.

Let ON is the shortest distance between AB and the line of action of P.

The force P can be considered to act at ON along its line of action. It can be resolved into two components:

i) $P \cos\theta$ parallel to AB and

ii) $P \sin\theta$ Perpendicular to it.

The component $P \cos\theta$ being $||^{lel}$ to AB has zero moment about AB.

The moment of $P \sin\theta$ about AB is $P \sin\theta$. ON and this in the moment of P about AB.

<u>Note: 1</u>

The moment of P about AB is zero if either (i) P is $||^{lel}$ to AB or else (ii). the line of action of P intersects AB.

<u>Note: 2</u>

As in the case of Varigon's theorem in two dimensions, we can show that, if a system of forces acting on a body have a resultant, the algebraic sum of their moments about any line in the body in equal to the moment of their resultant.

Note: 3

If a system of forces acting on a body keeps it in equilibrium, the algebraic sum of their moments about any line in the body is zero.

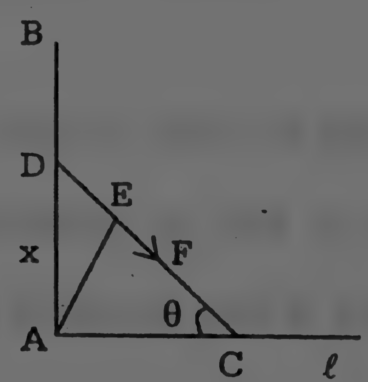
Example: 1

At what height from the base of a pillar must the end of a rope of a given length be fixed so that a man standing on the ground and pulling at its other end with a given force may have the greatest tendency to make the pillar overturn?

Solution:

Let AB be the pillar. A being the base and C the man of the ground.

Let the rope be fixed at D and AD = X. CD = ℓ = length of the rope.



Let F be the given force exerted by the man along DC. When the pillar overturns. The contact at A is broken. The man will have the best chance of overturning the pillar, if the moment of F about A is greatest.

Draw AE $\perp$ to DC and

Let $\angle DCA = \theta$

Moment of F about A

Space for Hints

Space for Hints

$= F.\ AE$

$= F.\ AC \text{ Sin } \theta$

$= F.\ DC \text{ Cos } \theta \text{ Sin } \theta \ (\Delta ADC)$

$= F\ \ell \text{ Cos } \theta \text{ Sin } \theta$

$$= \frac{F\ell}{2} 2\cos\theta\sin\theta = \frac{F\ell}{2}\text{Sin}2\theta$$

F and ℓ being given the above moment is greatest when Sin 2 θ is greates.

(ie) When $2\ \theta = 90^o$

Or $\theta = 45^o$

Then $X = AD = DC \text{ Sin } \theta$

$= \ell \text{ Sin } 45^o \ \frac{\ell}{\sqrt{2}}$ from the base.

Thus the rope is to be fixed at a height of $\frac{\ell}{\sqrt{2}}$ from the base.

Example: 2

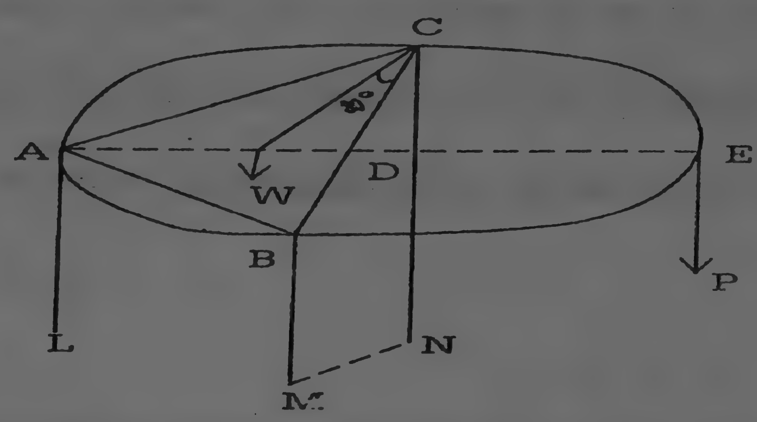
A round table of weight W stands on three legs of which the upper ends are attached to its rim so as to form an equilateral triangle show that a body whose weight does not exceed W may be piaced anywhere on the table without the risk of toppling it over.

Solution:

Let ABC be the round table in which A, B, C are the upper ends of the legs attached to the rim such that ABC is an equilateral triangle and let L,M,N be the points of contact of legs with the floor.

Leg G be the center of the table through which its weight W acts. There is a possibility of overturning if any weight P is placed on the portion of the

table outside the triangle ABC, say, in the portion, BEC.



If there is overturning the table will turn about the line MN and the let AL will rest very lightly on the floor.

In that case, the end L just loses contact with the floor and the weight P and W will have equal moments about MN.

(ie) About BC. The weight will have the greatest turning effect when placed farthest away from BC.

(ie) When placed at E, the midpoint of the arc BEC.

Let D be the midpoint of BC.

As Δ ABC in equilateral, AGDE is $\perp$ to BC.

Taking moments about BC.

We have

$$W.GD = P.\ DE$$

From right angled Δ GDC

$$\text{Sin } 30^\circ = \frac{GD}{GC}$$

$$GD = GC \text{ Sin } 30^\circ$$

$$\frac{GC}{2} = \frac{GE}{2} \text{ (G being the circumcentre of } \Delta \text{ ABC)}$$

D is the midpoint of GE and GD = DE

From (1) P = W.

Space for Hints

Hence W is the greatest weight that can be placed anywhere on the table without toppling it over. It can be noted that if any weight is placed within the Δ ABC, its moment about BC or CA or AB will be in the same direction as that of W and hence there is no change of the table being overturned whatever the weight may be.

Example: 3

A uniform circular plate is supported horizontally at three points A, B, C of its circumference. Show that the pressures on the supports are in the ratio Sin 2A: Sin 2B: Sin 2C.

Solution:

Let BC = a, CA = b and AB = c the weight of the plate acts at O. The centre of the circle and which is also the circumcentre of the triangle.

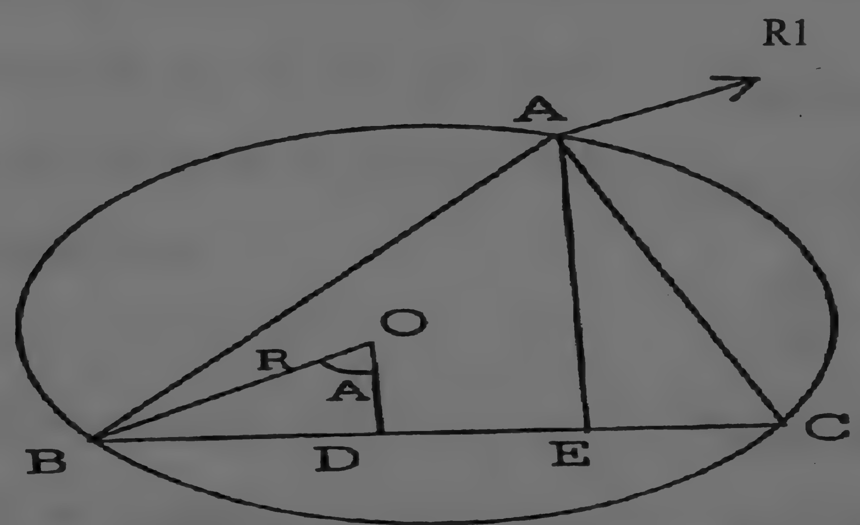
Let OD be perpendicular to BC.

We know that $\angle BOD = A$

From rt. $\angle$d Δ BOD

OD = OB. Cos $\angle BOD$ = R. Cos A

R being the circum – radius of the triangle.



Let AE be perpendicular to BC

AE = AC. Sin $\angle ACE$ = b Sin C

Let R_1 be the reaction at A.

Taking moments about BC, (to avoid the reactions at B and C)

We have at R_1. AE = W.OD

$$R_1 = \frac{W.RCosA}{b\sin C}$$

$$= \frac{WRCosA}{2RSinBSinC} \qquad (b = 2R \text{ Sin } B)$$

$$\frac{WCosA}{2SinB\sin C} = \frac{2W\sin ACosA}{4SinASinBSinC}$$

$$= \frac{W\sin 2A}{4SinASinBSinC}$$

Similarly the reactions, R_2 and R_3 at the other two supports are

$$\frac{WSin2B}{4SinASinBSinC} \text{ and } = \frac{WSin2C}{4SinASinBSinC}$$

$R_1 : R_2 :: R_3 :$ = Sin 2A: Sin 2B: Sin 2C.

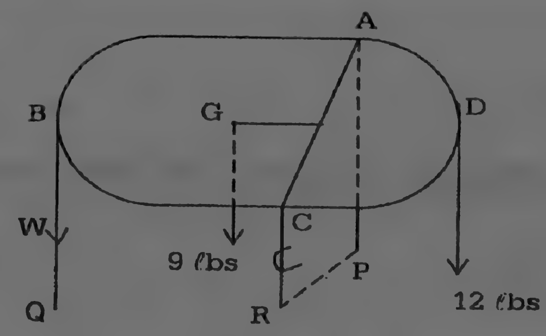
Example: 4

A table consists of a uniform circular board supported by three vertical legs fixed at equal distances around the circumference. The weight of the board is 9 ℓbs. A weight of 2 ℓbs. Placed on the edge of the table, midway between two legs is just sufficient to cause the table to over turn. Find the weight of each leg.

Solution:

The weight of the board is 9 ℓbs, and it acts through the centre G of the board vertically downwards.

Space for Hints



The weight 12 ℓbs in placed at D which in mid-way between the legs at A and C and this causes the table to overturn about the line PR joining the feet of the legs at A and C.

Let W be the weight of each leg.

Taking moments about the line PR (or AC)

We get 9 GE + W. EB = 12 ED.

If GD = r

The radius of circular board

$$EB = \frac{3}{2}r.$$

$$GE = ED = \frac{1}{2}r.$$

Hence $9.\frac{1}{2}r + w.\frac{3}{2} = 12.\frac{1}{2}r.$

W = 1 ℓb.

UNIT – 4

THREE FORCES ACTING ON A RIGID BODY

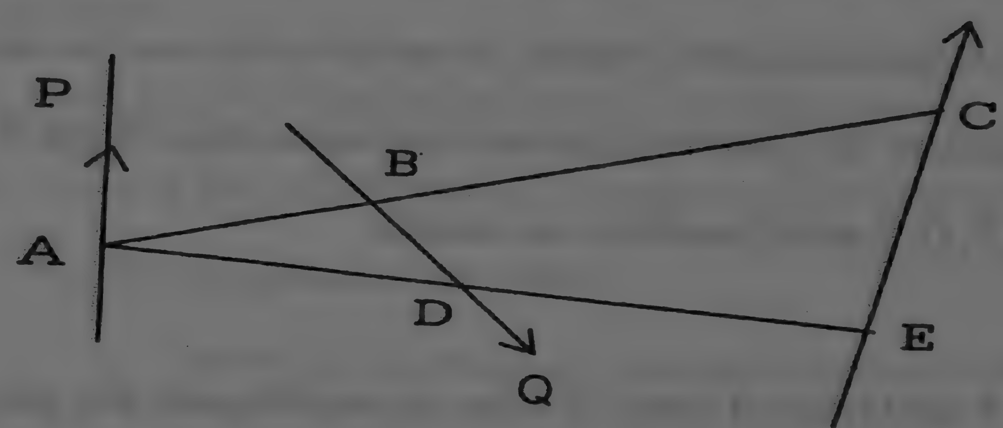
4.1 Rigid body subjected to any three forces:

There is a large class of problems in which a body is in equilibrium under the action of three forces. We shall first prove that if three forces acting on a rigid body are in equilibrium, they must be coplanar.

Let P, Q, R be three forces in equilibrium. Take any point A on the line of action of P and any point B on the line of action of Q such that AB is not parallel to R. Then the three forces being in equilibrium, the sum of their moments about the line AB is zero. But P and Q intersect AB and therefore their moments about AB are each zero.

$\therefore$ R is either parallel to AB or R intersects. AB.

But we have chosen the points A and B such that R is not parallel to AB.



R must intersect AB at a point, say C.

Similarly, if D is some other point on Q such that AD is not $||^{lel}$ to R.

We can prove that R must intersect AD also at a point, say E.

Since the lines BC and DE intersect at A. BD and CE must lie in one plane and A is on this plane.

(ie) A is a point on the plane formed by Q and R.

But A is any point on the line of action of P.

Space for Hints

Every point on P is a point on the plane formed by Q and R.

(ie) P, Q, R are in one plane.

4.2 Three Coplanar Forces:

Theorem:

If three coplanar forces acting on a rigid body keep it in equilibrium, they must either be concurrent or be all parallel.

Proof:

Let P, Q, R be three coplanar forces acting on a body and keep it in equilibrium.

Then R must be equal and opposite to the resultant of P and Q.

Now, P and Q being coplanar must either be parallel or intersect.

Case: 1

If P and Q are parallel (like or unlike), their resultant is also a parallel force. As R balances the above resultant, it must act in the same line but in opposite direction. So R also is in the same direction as that of P and Q.

(ie) P, Q, R are all parallel to one another.

Case: 2

Let P and Q meet at a point O. Then, by parallelogram law, their resultant is a force through O. As this is balanced by the third force R, the line of action of R must also pass through O.

(ie) the three forces are concurrent.

Note:

In the above discussion P and Q can never form a couple. Since we know that a couple and a force can never be in equilibrium.

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4.3 Conditions of Equilibrium:

When the number of forces acting on rigid body in equilibrium is three and when the forces are not parallel, we can use the methods which apply to forces acting on a particle. Thus we can use Lami's theorem, or the triangle of forces or we can resolve the forces is two directions at rigid angles to each other.

When the three force in equilibrium are parallel, we use the condition that each is proportional to the distance between the other two.

In all cases, it is important to draw a figure with the three forces clearly shown, either all parallel or meeting in a point.

Procedure to be followed in solving any statical:

In solving any statical problem the student should proceed in the following mannar:

1. First draw the figure according to the conditions given.
2. Mark all the forces acting on the body or bodies, bearing in mid the following fundamental points:
    a. The weight of a body acts vertically downwards though its centre of gravity.
    b. When a body is leaning against a smooth surface, the reaction of the body is normal to the surface.
    c. When a rod is resting on a smooth peg, the reaction of the peg on the rod is perpendicular to the rod.
    d. The tension is a light string is the same throughout its length and this tension is unaffected by the string passing over smooth pegs or pulleys. If the pulley is rough, the tension is different on the two sides of the pulley.

Space for Hints

e. The resultant of two equal forces bisects the angle between them.

In addition to the above considerations, we can use the fact that when there are only three non-parallel forces, they must meet in a point. Thus it three forces are in equilibrium and two of them meet at a point O, the third also must pass through O. This consideration will enable us to draw an accurate figure slowing the position of the body.

4.4 Two trigonometrical theorems:

The following two important trigonometrical theorems will be found to be highly useful in the solution of many statical problems:

If D is any point on the base BC of triangle ABC such that $\frac{BD}{DC} = \frac{m}{n}$ and

$\angle ADC = \theta$, $\angle BAD = \alpha$ and $\angle DAC = \beta$,

then $(m+n)\ \text{Cot}\ \theta = m\text{Cot}\ \alpha - n\text{Cot}\ \beta$ (1)

$(m+n)\ \text{Cot}\ \theta = n\text{Cot}\ B - m\ \text{Cot}\ C$ (2)

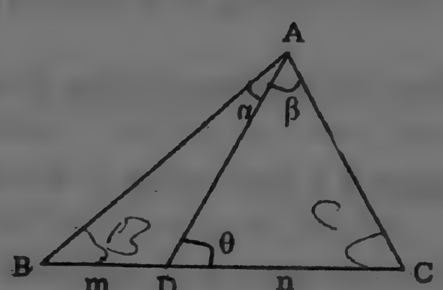
Proof:

$$(1)\ \frac{m}{n} = \frac{BD}{DC} = \frac{BD}{DA} \cdot \frac{DA}{DC}$$

$$= \frac{\text{Sin}\angle BAD}{\text{Sin}\angle ABD} \cdot \frac{\text{Sin}\angle ACD}{\text{Sin}\angle DAC}$$

$$= \frac{\text{Sin}\alpha}{\text{Sin}(\theta-\alpha)} \text{X} \frac{\text{Sin}(\theta+\beta)}{\text{Sin}\beta}$$

$[(\because \angle ACD = 18 - (\theta+\beta))]$



$$= \frac{Sin\alpha(Sin\theta Cos\beta + Cos\theta Sin\beta)}{Sin\beta(Sin\theta Cos\alpha Cos\theta)}$$

$$= \frac{Cot\beta + Cot\theta}{Cot\alpha - Cot\theta}$$

(dividing the numerator and denominator by Sin α Sin β Sin θ)

$\therefore n\,((Cot\beta + Cot\theta) = m(Cot\alpha - Cot\theta))$

$(m + n)$ Cot $\theta = m\cot\alpha - n\cot\beta$

(2) Again

$$\frac{m}{n} = \frac{Sin\angle BAD}{Sin\angle ABD}.\frac{Sin\angle ACD}{Sin\angle DAC}$$

$$= \frac{Sin(\theta - B).\sin C}{SinBSin(C+\theta)} \quad (\because DAC = 180^o - \overline{\theta + C})$$

$$= \frac{SinC(Sin\theta CosB - Cos\theta SinB)}{SinB(SinCCos\theta + CosCSin\theta)}$$

$$= \frac{CotB - Cot\theta}{Cot\theta + CotC}$$

(Dividing the numerator and the denominator by Sin B Sin C Sin θ)

(ie) $m\ (\cot\theta + CotC) = n(CotB - Cot\theta)$

(or) $(m + n)$ Cot $\theta = nCotB - mCotC$.

Example: 1

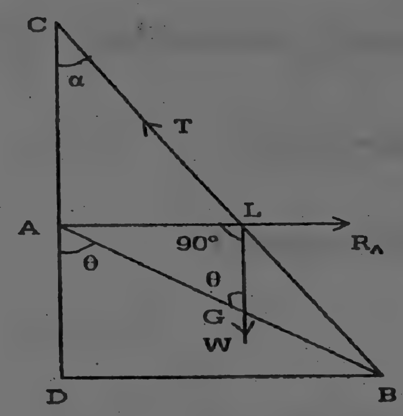
A uniform rod of length a hangs against a smooth vertical wall being supported by means of a string, of length ℓ, tied to one end of the rod, the other end of the string attached to a point in the wall. Show that the rod can rest inclined to the wall at an angle θ given by $Cos^2\ \theta = \frac{\ell^2 - a^2}{3a^2}$

What are the limits of the ratio of a: ℓ in order that equilibrium may be possible?

Space for Hints

Solution:

AB is the rod of length a with G its centre of gravity and BC is the string of length ℓ.



The forces acting on the rod are:

i) its weight W acting vertically downwards through G.

ii) the reaction R_A at A which is normal to the wall and therefore horizontal.

iii) the tension T of the string along BC.

These three forces in equilibrium not being all parallel, must meet in a point L, as shown in the figure.

Let the string make an angle α with the vertical.

$\angle ACB = \alpha = \angle GLB$

Also $\angle LGB = 180^\circ - \theta$ and $\angle ALG = 90^\circ$

Using the first trigonometrical theorem

Δ ALB and noting that AG: GB = 1:1

We have

$$(1+1) \text{ Cot } (180^\circ - \theta) = 1. \text{ Cot } 90^\circ - 1. \text{ Cot } \alpha$$

$$- 2 \text{ Cot } \theta = - \text{ Cot } \alpha$$

$$\text{Or } 2 \text{ Cot } \theta = \text{Cot } \alpha \qquad (1)$$

Draw BD $\perp$ to CA

From rt. <d Δ CDB, BD = BC. Sin α and

From rt. <d Δ ABD, BD = AB Sin θ a Sin θ

$\ell \sin \alpha = a \operatorname{Sin} \theta$

(2)

Eliminate α between (1) and (2)

We know that $\operatorname{cosec}^2 \alpha = 1 + \operatorname{Cot}^2 \alpha$

(3)

From (2),

$$\operatorname{Sin} \alpha = \frac{a\operatorname{Sin}\theta}{\ell}$$

$$\operatorname{cosec} \alpha = \frac{\ell}{a\operatorname{Sin}\theta}$$

(4)

Substituting (4) and (1) in (3),

We have

$$\frac{\ell^2}{a^2\operatorname{Sin}^2\theta} = 1 + 4\operatorname{Cot}^2\theta \Rightarrow \frac{\ell^2}{a^2\operatorname{Sin}^2\theta} = 1 + 4\frac{\operatorname{Cos}^2\theta}{\operatorname{Sin}^2\theta}$$

$$\text{(ie)}\ \frac{\ell^2}{a^2} = \operatorname{Sin}^2\theta + 4\operatorname{Cos}^2\theta$$

$$\frac{\ell^2}{a^2} = \operatorname{Sin}^2\theta + \operatorname{Cos}^2\theta + 3\operatorname{Cos}^2\theta$$

$$3\operatorname{Cos}^2\theta = \frac{\ell^2}{a^2} - 1$$

$$= \frac{\ell^2 - a^2}{a^2}$$

$$\operatorname{Cos}^2\theta = \frac{\ell^2 - a^2}{3a^2}$$

For the above equilibrium position to be possible, $\operatorname{Cos}^2 \theta$ must be

Space for Hints

positive and less then 1.

$$\ell^2 - a^2 > 0.$$

(ie) $\ell^2 > a^2$ or $a^2 < \ell^2$

Also $\dfrac{\ell^2 - a^2}{3a^2} < 1$

(ie) $\ell^2 - a^2 < 3a^2$ or $\ell^2 < 4a^2$

(ie) $a^2 > \dfrac{\ell^2}{4}$

a^2 lies between $\dfrac{\ell^2}{4}$ and ℓ^2.

(ie) $\dfrac{a^2}{\ell^2}$ lies between $\dfrac{1}{4}$ and 1.

(or) $\dfrac{a}{\ell}$ must lie between $\dfrac{1}{2}$ and 1.

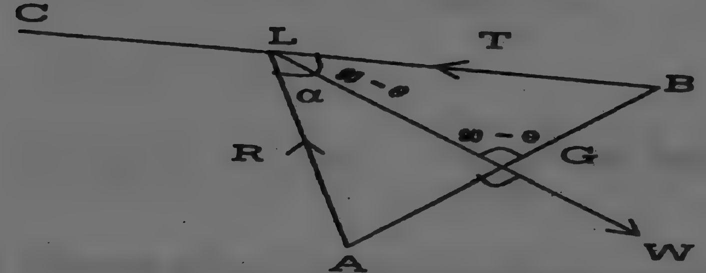
Example: 2

A beam of weight W hinged at one end is supported at the other end by a string so that the beam and the string are in a vertical plane and make the same angle θ with the horizon. Show that reaction at the hinge is

$$\frac{W}{4}\sqrt{8 + \operatorname{cosec}^2\theta}$$

Solution:

Let AB be the beam of weight W and G its centre of gravity. BC is the string.



Space for Hints

The forces acting on the beam are:

i) its weight W acting vertically downwards at G.

ii) the tension T along BC and

iii) The reaction R at the hinge at A.

Let the forces (i) and (ii) meet at L.

For equilibrium, the third force R also must pass through L.

(ie) the reaction at the hinge is a force along AL.

BC and AB make the same angle θ with the horizon.

They make the same angle $90^\circ - \theta$ with the vertical LG.

(ie) $\angle BLG = 90^\circ - \theta = \angle LGB$

Let $\angle ALG = \alpha$

Using the first trigonometrical theorem Δ ALB

We have (as AG: GB = 1:1)

$(1 + 1)\,\mathrm{Cot}\,(90 - \theta) = 1.\,\mathrm{Cot}\,\alpha - 1.\mathrm{Cot}(90^\circ - \theta)$

(ie) $2\tan\theta = \mathrm{Cot}\alpha - \tan\theta$

(or) $3\tan\theta = \mathrm{Cot}\alpha$ (1)

Applying Lami's theorem for the three forces at L,

We have $\dfrac{R}{\mathrm{Sin}(90^\circ - \theta)} = \dfrac{W}{\mathrm{Sin}(90^\circ - \theta + \alpha)}$

(ie) $\dfrac{R}{\mathrm{Cos}\theta} = \dfrac{W}{\mathrm{Sin}(90^\circ - \overline{\theta - \alpha})} = \dfrac{W}{\mathrm{Cos}(\theta - \alpha)}$

$$R = \frac{W\mathrm{Cos}\theta}{\mathrm{Cos}(\theta - \alpha)} = \frac{W\mathrm{Cos}\theta}{\mathrm{Cos}\theta\mathrm{Cos}\alpha + \mathrm{Sin}\theta\mathrm{Sin}\alpha}$$

$$= \frac{W\mathrm{Cos}\theta}{\mathrm{Sin}\alpha(\mathrm{Cos}\theta\mathrm{Cot}\alpha + \mathrm{Sin}\theta)}$$

$$= \frac{W\cos\theta}{\mathrm{Sin}\alpha(\mathrm{Cos}\theta.3\tan\theta + \mathrm{Sin}\theta)} \text{ using (1)}$$

$$= \frac{W\cos\theta\,\mathrm{cosec}\,\alpha}{3\mathrm{Sin}\theta + \mathrm{Sin}\theta}$$

$$= \frac{W\mathrm{Cos}\theta\,\mathrm{cosec}\,\alpha}{4\mathrm{Sin}\theta}$$

$$= \frac{w}{4}\mathrm{Cot}\theta\,\mathrm{cosec}\,\alpha$$

$$= \frac{w}{4}\mathrm{Cot}\theta\sqrt{1+\mathrm{Cot}^2\alpha}$$

$$= \frac{W\mathrm{Cot}\theta}{4}\sqrt{1+9\tan^2\alpha} \qquad [\because \text{by } (1)]$$

$$= \frac{W}{4}\sqrt{\mathrm{Cot}^2\theta+9} = \frac{W}{4}\sqrt{\mathrm{Cot}^2\theta+1+8}$$

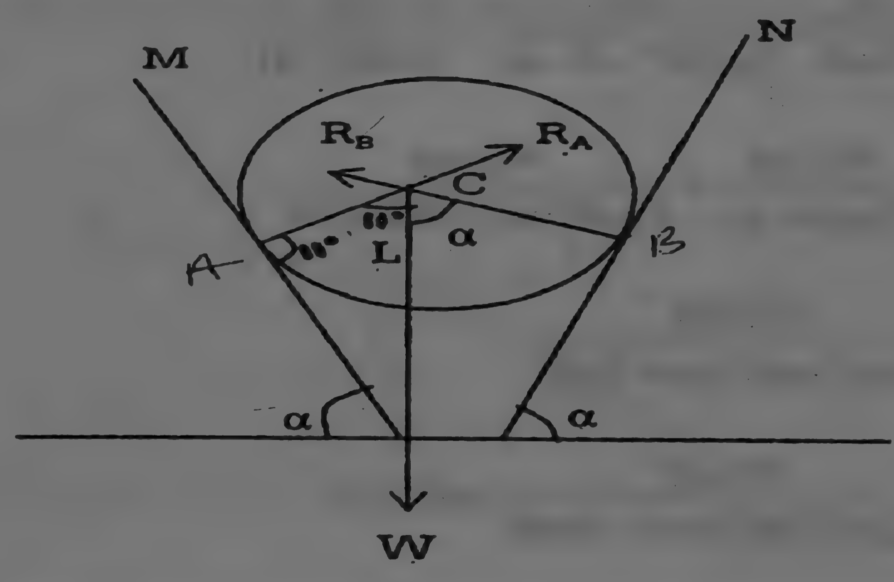
$$R = \frac{W}{4}\sqrt{\mathrm{Cosec}^2\theta+8}$$

Example: 3

A heavy uniform sphere rests touching two smooth inclined planes one of which is inclined at 60^o to the horizontal. If the pressure on this plane is one-half of the weight of the sphere, prove that the inclination of the other plane to the horizontal is 30^o.

Solution:

Let the sphere centre C rest on the inclined planes AM and BN. MA makes 60^o with the horizontal and let NB make an angle α with the horizon.



Space for Hints

The forces acting on the sphere are

i) Reaction R_A at A $\perp^r$ to the inclined plane AM and to the sphere and hence passing through C.

ii) Reaction R_B at B which is normal to the inclined plane BN and to the sphere and hence passing through C.

iii) W the weight of the sphere acting vertically downwards at C along CL.

Clearly the above three forces meet at C.

Also $\angle ACL = 60^\circ$

And $\angle BCL = \alpha$

Applying Lami's theorem,

$$\frac{R_A}{\text{Sin}\alpha} = \frac{W}{\text{Sin}(60^\circ + \alpha)}$$

$$R_A = \frac{W\text{Sin}\alpha}{\text{Sin}(60^\circ + \alpha)}$$

But $R_A = \dfrac{W}{2}$ (2)

From (1) and (2)

We have

$$\frac{W\text{Sin}\alpha}{\text{Sin}(60^\circ + \alpha)} = \frac{W}{2}$$

(ie) $2\text{Sin}\alpha = \text{Sin}(60^\circ + \alpha)$

$= \text{Sin } 60^\circ \text{ Cos } \alpha + \text{Cos}60^\circ \text{Sin}\alpha$

(ie) $2\text{Sin}\alpha = \dfrac{\sqrt{3}}{2}\text{Cos}\alpha + \dfrac{1}{2}\text{Sin}\alpha$

(ie) $4\text{Sin}\alpha = \sqrt{3}\text{Cos}\alpha + \text{Sin}\alpha$

(ie) $3\text{Sin}\alpha = \sqrt{3}\text{Cos}\alpha$

(ie) $\dfrac{\operatorname{Sin}\alpha}{\operatorname{Cos}\alpha} = \dfrac{\sqrt{3}}{3}$

(ie) $\tan\alpha = \dfrac{1}{\sqrt{3}}$

(or) $\alpha = 30^{\circ}$

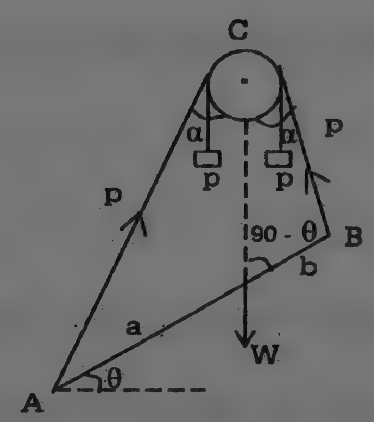
Example: 4

Equal weights P and P are attached to two strings ACP and BCP passing over a smooth peg C. AB is heavy beam of weight W, whose centre of gravity is 'a' ft from A and 'b' ft from B. Show that AB is inclined to the horizon at an angle

$$\tan^{-1}\left[\frac{a-b}{a+b}.\tan(\operatorname{Sin}^{-1}\frac{W}{2P})\right]$$

Solution:

Since the string ACP and BCP support the equal weights P at their ends the tensions along AC and BC must be each equal to P.



The forces acting on the rod are

i) tension P along AC

ii) Tension P along BC and

iii) Its weight W acting vertically downwords through G, its centre of gravity.

For equilibrium, the above three forces must meet at a point C as in the figure.

Space for Hints

Now the resultant of the two equal tensions P along AC and CB must balance the weight W.

Hence CG must bisect the angle ACB.

Let $\angle ACG = \alpha = \angle BCG$.

Then $W = 2P \cos \alpha$ (1)

Let AB make an angle θ with the horizontal.

Then $\angle CGB = 90^\circ - \theta$.

Using the first trigonometrical theorem.

to Δ ACB.

We have $(a + b) \cot (90^\circ - \theta) = a\cot\alpha - b\cot\alpha$

(ie) $(a + b) \tan \theta = (a - b)\cot\alpha$

$$\tan\theta = \frac{a-b}{a+b}\cot\alpha$$

$$= \frac{a-b}{a+b}\tan(\frac{\pi}{2}-\alpha)$$

$$= \frac{a-b}{a+b}\tan\left(\frac{\pi}{2}-\cos^{-1}\left(\frac{W}{2P}\right)\right)$$

[$\because$ by (1)]

$$= \frac{a-b}{a+b}\tan(\sin^{-1}\frac{W}{2P})$$

$(\because \sin^{-1}x+\cos^{-1}x = \frac{\pi}{2})$

$$\theta = \tan^{-1}(\frac{a-b}{a+b}\tan(\sin^{-1}\frac{W}{2P}))$$

Example: 5

The attitude of a right cone is h and the radius of its base is a, A string is fastened to the vertex and to a point on the circumference of the

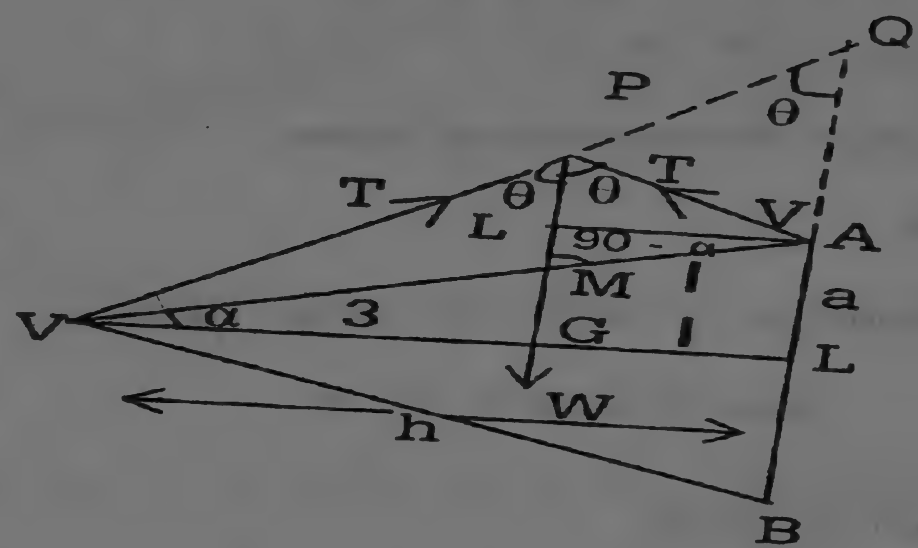
circular base and is then put over a smooth peg. The cone rests with its axis horizontal. Show that the length of the string is $\sqrt{h^2 + 4a^2}$

Solution:

VAB is the cone resting with the axis VL horizontal.

G is its centre of graivity.

We know that $\dfrac{VG}{GL} = \dfrac{1}{3}$



Since the string passes over a smooth peg P, the tensions along VP and PA must be equal. Let them T each. Since the two tensions pass through P. for equilibrium, the third force.

(ie) W, the weight of the cone must pass through P.

Hence PG is vertical.

Since W balances the resultant of the equal tensions T, the line PG must bisect $\angle VPA$.

Let $\angle VPG = \theta = \angle APG$

Let α be the semi-vertical angle of the cone and let PG meet VA at M.

$\angle PMA = 90^o - \alpha$

$$\frac{VM}{MA} = \frac{VG}{GL} = \frac{3}{1}$$

Space for Hints

Using the first trigonometrical theorem.

Δ VPA

We have

$(3 + 1)\ \text{Cot}\ (90^\circ - \alpha) = 3\ \text{Cot}\ \theta - 1.\ \text{Cot}\ \theta$

(ie) $4\tan\alpha = 2\ \text{Cot}\ \theta$ (1)

(or) $2\tan\alpha = \text{Cot}\ \theta$

From Δ PVG,

$$\text{Sin}\ \theta = \frac{VG}{VP}$$

$$VP = \frac{VG}{\text{Sin}\theta} \quad (2)$$

From Δ PNA,

$$\text{Sin}\ \theta = \frac{AN}{PA}$$

$$PA = \frac{AN}{\text{Sin}\theta} = \frac{GL}{\text{Sin}\theta}$$

(3)

Adding (2) & (3)

$$VP + PA = \frac{VG}{\text{Sin}\theta} + \frac{GL}{\text{Sin}\theta}$$

$$= \frac{VG + GL}{\text{Sin}\theta} = \frac{h}{\text{Sin}\theta}$$

Length of the string $= \dfrac{h}{\text{Sin}\theta}$

$= h\ \text{cosec}\ \theta$

$= h\sqrt{(1 + \text{Cot}^2\theta)}$

$(\because \text{Cosec}^2\theta = 1 + \text{Cot}^2\theta)$

$= h\sqrt{1 + 4\tan^2\alpha}$ using (1)

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$$= h\sqrt{1+\frac{4a^2}{h^2}}$$

$$(\because \tan\alpha = \frac{AL}{VL} = \frac{a}{h})$$

$$= \sqrt{h^2+4a^2}$$

Aliter:

Produce VP to meet BA produced at O. $\angle PAQ = \theta$ and $\angle PQA = \theta$ (PG||AB)

$\therefore \angle PAQ = \angle PQA$ and PQ = PA

Hence length of the string

= VP + PA = VP + PQ = VQ

From (1) Cot $\theta = 2\tan\alpha$

$$= \frac{2a}{h}$$

$$\tan\theta = \frac{h}{2a} \qquad (4)$$

From rt. $\angle$d ΔVQL

$$\tan\ \theta = \frac{VL}{QL} = \frac{h}{QL} \qquad (5)$$

From (4) and (5) QL = 2a

Now, $VQ^2 = VL^2 + QL^2$

$VQ^2 = h^2 + 4a^2$

$VQ^2 = \sqrt{h^2+4a^2}$

Example: 6

A string of length 2ℓ has one end attached to the extremity of a smooth heavy uniform rod of length 2a and the other caries a weightless ring which slides on the rod. The rod is suspended by means of the string from a smooth

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peg, prove that, if θ be the angle which the rod makes with the vertical, then $\ell \cos\theta = a \sin^3\theta$.

Solution:

Let AB be the rod. G its centre of gravity, C the peg and D the ring. The forces acting on the ring are

i) tension T along DC and

ii) the normal reaction R at D due to contact with rod, which is $\perp^r$ to AB downwards.

Hence for equilibrium of the ring, T = R.

Also, DC is $\perp^r$ to AB.

Since the string passes over a smooth peg, the tensions along DC and AC must be equal. Let them be T each.

Consider the equilibrium of the rod and the ring together.

The forces acting on them are:-

i) tension T along AC

ii) tension T along DC and

iii) Weight W of the rod acting vertically downwards through G.

The mutual normal reactions cancel each other As the forces (i) and (ii) pass through C for equilibrium, W also must pass through C.

$\therefore$ CG in Vertical

Since W balances the resultant of the equal tensions T, the line CG must bisect $\angle ACD$.

$\angle ACG = \angle GCD = 90^\circ - \theta$

Using the first trigonometrical equation to Δ ACD

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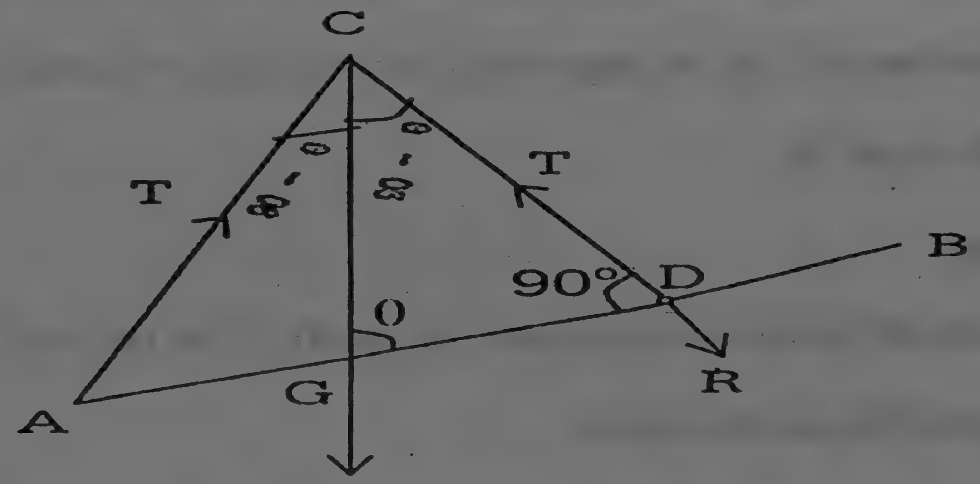

We have

$$(AG + GD) \text{ Cot } \theta = AG. \text{ Cot } (90^o - \theta) - GD. \text{ Cot } (90^o - \theta)$$

(ie) $(a + G.D) \text{ Cot } \theta = \tan \theta - GD \tan \theta$

$$GD (\text{Cot } \theta + \tan \theta) = a (\tan \theta - \text{Cot } \theta)$$

$$GD = \frac{a(\tan\theta - \text{Cot}\theta)}{(\text{Cot}\theta + \tan\theta)}$$

$$= a \frac{\frac{\text{Sin}\theta}{\text{Cos}\theta} - \frac{\text{Cos}\theta}{\text{Sin}\theta}}{\frac{\text{Cos}\theta}{\text{Sin}\theta} + \frac{\text{Sin}\theta}{\text{Cos}\theta}}$$

$$= \frac{a(\text{Sin}^2\theta - \text{Cos}^2\theta)}{\text{Cos}\theta\text{Sin}\theta} \frac{\text{Sin}\theta\text{Cos}\theta}{(\text{Cos}^2\theta + \text{Sin}^2\theta)}$$

$$= a (\text{Sin}^2\theta - \text{Cos}^2\theta) \qquad (1)$$

From Δ ACG,

$$\frac{AC}{\text{Sin}(180^o - \theta)} = \frac{AG}{\text{Sin}(90^o - \theta)}$$

$$\text{(ie)} \ \frac{AC}{\text{Sin}\theta} = \frac{a}{\text{Cos}\theta} \text{ or } AC = \frac{a\text{Sin}\theta}{\text{Cos}\theta} \qquad (2)$$

From Δ GCD

$$\frac{CD}{\text{Sin}\theta} = \frac{GD}{\text{Sin}(90^o - \theta)}$$

$$CD = GD \frac{\text{Sin}\theta}{\text{Cos}\theta} = a(\text{Sin}^2\theta - \text{Cos}^2\theta)\frac{\text{Sin}\theta}{\text{Cos}\theta} \qquad (3)$$

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(Using (1))

Adding (2) and (3)

$$AC + CD = \frac{a\sin\theta}{\cos\theta} + \frac{a(\sin^2\theta - \cos^2\theta)}{\cos\theta}\sin\theta$$

$$\text{(ie) } 2\ell = \frac{a\sin\theta}{\cos\theta}\,(1 + \sin^2\theta - \cos^2\theta)$$

$$2\ell = \frac{2a\sin^3\theta}{\cos\theta}$$

(or) $\ell \cos\theta = a\sin^3\theta$

4.5 Problems on parallel forces:

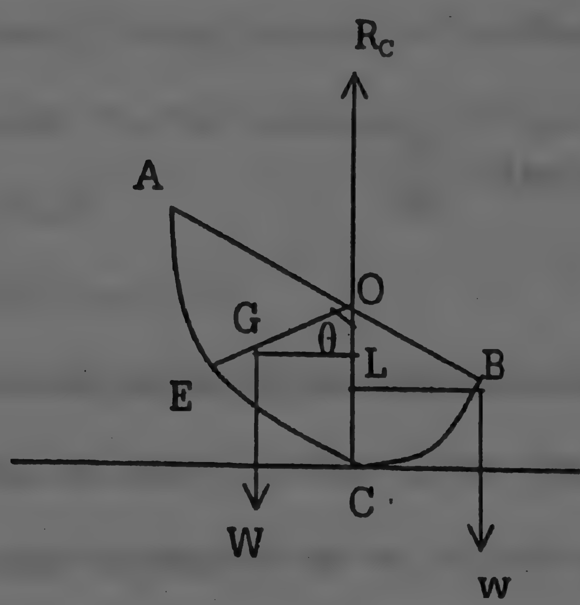
Example: 1

A uniform solid hemisphere of weight W rests with its curved surface on a smooth horizontal plane. A weight w its suspended form point on the rim of the hemisphere. If the plane base of the rim is inclined to the horizontal at an angle θ,

Prove that $\tan\theta = \frac{8w}{3W}$

Solution:

The forces acting on the hemisphere are



i) the reaction R_C at C $\perp^r$ to the horizontal plane and to the

hemisphere. Hence it is along CO, O being the geometric centre of the hemisphere.

ii) The weight W of the hemisphere acting vertically downwards at G, the centre of gravity of the body, which is on the radius OE $\perp^{r}$ to AB such that

$OG = \frac{3r}{8}$, r being the **radius**.

iii) the weight W **acting vertically** downwards at B. These three forces are clearly parallel.

For equilibrium, we know that each of the three forces is proportional to the distance between the other two.

$$\frac{W}{BD} = \frac{w}{GL} \qquad (1)$$

From the figure. $BD = OB.\ \mathrm{Cos}\theta = r\mathrm{Cos}\theta$ and

$$GL = OG.\ \mathrm{Sin}\,\theta$$

$$= \frac{3r}{8}\mathrm{Sin}\theta$$

(1) becomes

$$\frac{W}{r\mathrm{Cos}\theta} = \frac{w}{(\frac{3r}{8})\mathrm{Sin}\theta}$$

$$\text{(or) Sin}\frac{\mathrm{Sin}\theta}{\mathrm{Cos}\theta} = \frac{8w}{3W} \Rightarrow \tan\theta = \frac{8w}{3W}$$

Example: 2

A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ, ϕ are the inclinations of the string and the plane

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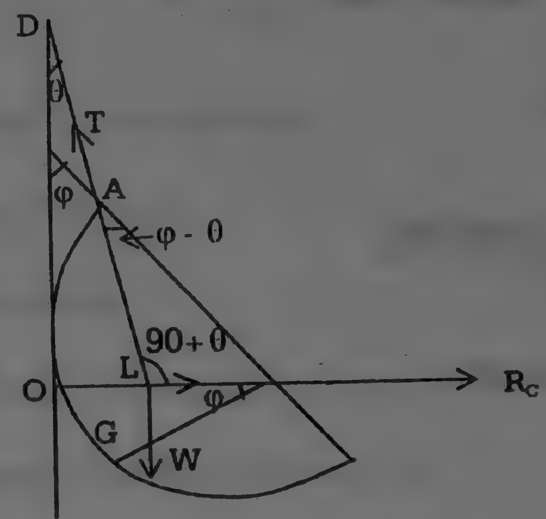
base of the hemisphere to the vertical, prove that $\tan\phi = \frac{3}{8} + \tan\theta$.

Solution:

The figure represents a section of the hemisphere and the wall by a plane $\perp$ to the wall and the base of the hemisphere. AB is the plane base and O is the centre of the hemisphere C is the point of contact with the wall.



We know that, G the centre of gravity of the hemisphere is on the radius $\perp^r$ to AB such that $OG = \frac{3r}{8}$.

The forces acting on the hemisphere are:

i) the reaction R_C at C perpendicular to the wall and the hemisphere and hence passing through O.

ii) the tension T along the string AD and

iii) the weight W of the hemisphere acting at G vertically downwards.

Let the forces (i) and (ii) meet at L.

For equilibrium, the force (iii) also must pass through L.

LG is vertical.

$\angle LAO = \phi - \theta$

$\angle ALO = 90^\circ + \theta$

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$LOG = \phi$

From rt <d. Δ OLG. $OL = OG.\ Cos\ \phi = \frac{3r}{8} Cos\phi$ (1)

From Δ OAL, $\frac{OL}{Sin\angle OAL} = \frac{OA}{Sin\angle OLA}$

$$\text{(ie)} \ \frac{OL}{Sin(\phi - \theta)} = \frac{r}{Sin(90^\circ + \theta)}$$

$$OL = \frac{rSin(\phi - \theta)}{Cos\theta}$$

$$= \frac{r(Sin\phi Cos\theta - Cos\phi Sin\theta)}{Cos\theta}$$

$$= r\,(Sin\phi - Cos\phi \tan\theta) \qquad (2)$$

Equating (1) and (2)

$$\frac{3r}{8}\cos\phi = r(Sin\phi - Cos\phi\tan\theta)$$

Dividing by r Cos ϕ

We have

$$\frac{3}{8} = \tan\phi - \tan\theta$$

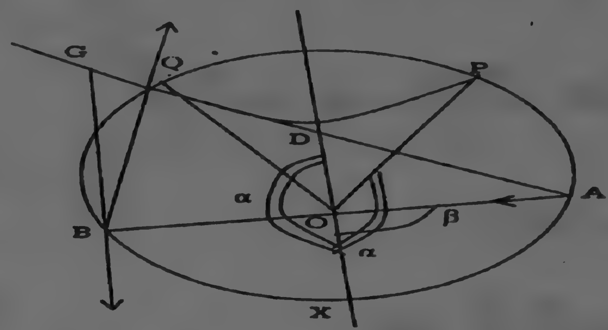
$$\text{(or) } \tan\ \phi = \frac{3}{8} + \tan\theta$$

Example: 3

A bowl is formed form a hollow sphere, of radius 'a' and is so placed that the radius of the sphere, drawn to each point in the rim makes an angle α with the vertical, whilst the radius drawn to a point A of the bowl makes an angle β with the vertical: if a smooth uniform rod remains at rest with one end at A and a point of its length in contact with the rim, show that the length of the rod is $4a \sin\beta$. Sec $\frac{\alpha - \beta}{2}$.

Solution:

PQ is the rim of the bowl and $\angle POX = \alpha = \angle QOX$



$\angle AOX = \beta . \angle AOB$ is the diameter.

G is the centre of gravity of the rod.

The forces acting on the rod are:

i) the reaction at A normal to the spherical surface and which hence passes through O the centre.

ii) The reaction at Q $\perp^r$ to the rod AG.

Since $\angle AQB = 90^\circ$

(AB being a diameter this reaction passes through B.)

iii) The weight of the rod acting vertically downwords through G.

The forces (i) and (ii) Pass through B.

For equilibrium, the force (iii) also must pass through B and

Hence GB is vertical

$\angle BOX = \angle 180^\circ - \beta$

$\angle BOQ = \angle QOX - \angle BOX = \alpha - (180^\circ - \beta)$

$$= \alpha + \beta - 180^\circ$$

$\angle BAQ = \frac{1}{2} \angle BOQ = \frac{1}{2}(\alpha + \beta - 180^\circ) = \frac{\alpha+\beta}{2} - 90^\circ$

From rt. $\angle$d Δ BAQ,

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$$QA = AB.\ \text{Cos}\ \angle BAQ$$

$$= 2a\ \text{Cos}\ (\frac{\alpha+\beta}{2} - 90^\circ)$$

$$= 2a\ \text{Cos}\ (90^\circ - \frac{\alpha+\beta}{2}) = 2a\ \ \text{Sin}\ \frac{\alpha+\beta}{2} \qquad (1)$$

$$QB = AB\ \text{Sin}\ \angle BAQ$$

$$= 2a\ \text{Sin}\ (\frac{\alpha+\beta}{2} - 90^\circ)$$

$$= 2a\ \text{Sin}\ [-(90^\circ - \frac{\alpha+\beta}{2})]$$

$$= -2a\ \text{Sin}\ [90 - \frac{\alpha+\beta}{2}]$$

$$= -2a\ \text{Cos}\ \left(\frac{\alpha+\beta}{2}\right) \qquad (2)$$

Let the vertical line OX cut the rod AG at D

$$\angle DOA = 180^\circ - \angle AOX = 180^\circ - \beta$$

$$\angle OAD = \angle BAQ = \frac{\alpha+\beta}{2} - 90^\circ$$

$$\angle ODA = 180^\circ - (\angle DOA + \angle OAD)$$

$$= 180^\circ - (180^\circ - \beta + \frac{\alpha+\beta}{2} - 90^\circ)$$

$$= \beta - \frac{\alpha-\beta}{2} + 90^\circ$$

$$= 90^\circ + \frac{\beta-\alpha}{2}$$

Since GB in $||^{lel}$ to OX.

$$\angle BGQ = \angle ODA = 90^\circ + \frac{\beta-\alpha}{2}$$

From rt $\angle$d Δ BGQ

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$$\tan \angle BGQ = \frac{BQ}{GQ}$$

$$\text{(ie) } \tan\left(90^\circ + \frac{\beta-\alpha}{2}\right) = \frac{BQ}{GQ}$$

$$\text{(ie) } \tan\left(90^\circ - \frac{\alpha-\beta}{2}\right) = \frac{BQ}{GQ}$$

$$\text{(or) } \text{Cot}\,\frac{\alpha-\beta}{2} = \frac{BQ}{GQ}$$

$$GQ = \frac{BQ}{\text{Cot}\frac{\alpha-\beta}{2}}$$

$$= -\frac{2a\text{Cos}\frac{\alpha+\beta}{2}}{\text{Cot}\frac{\alpha-\beta}{2}} \qquad \text{------ (3) using (2)}$$

$$AG = AQ + QG$$

$$= 2a\,\text{Sin}\,\frac{\alpha+\beta}{2} - \frac{2a\text{Cos}\frac{\alpha+\beta}{2}}{\text{Cot}\frac{\alpha-\beta}{2}} \qquad \text{(from (1) and (3))}$$

$$= 2a\,\text{Sin}\,\frac{\alpha+\beta}{2} - \frac{2a\text{Cos}\frac{\alpha+\beta}{2}\text{Sin}\frac{\alpha-\beta}{2}}{\text{Cos}\frac{\alpha-\beta}{2}}$$

$$= \frac{2a}{\text{Cos}\frac{\alpha-\beta}{2}}\left[\text{Sin}\frac{\alpha+\beta}{2}\text{Cos}\frac{\alpha-\beta}{2} - \text{Cos}\frac{\alpha+\beta}{2}\text{Sin}\frac{\alpha-\beta}{2}\right]$$

$$= \frac{2a}{\text{Cos}\frac{\alpha-\beta}{2}}\text{Sin}\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right)$$

$$= 2a\,\text{Sec}\,\frac{\alpha-\beta}{2}\text{Sin}\beta$$

Length of the rod = 2 AG = 4 a Sec $\frac{\alpha-\beta}{2}\text{Sin}\beta$.

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UNIT – 5

FRICTION

5.1 Introduction:

When two bodies are in contact, with their surfaces of contact smooth one body in free to move on the other along the plane of contact this means that the surfaces cannot provide any force preventing motion along the plane of contact. (ie) the reaction at the point of contact is just normal to the plane of contact. However, if the surfaces are rough, a force is called into play along the plane of contact preventing motion along the plane. Such a tangential force is called the frictional force. This force is just the component of the reaction at the point of contact along the plane of contact (the other component, $\perp^r$ to the plane being called normal reaction).

5.1.1 Definition:

If two bodies are in contact with one another, the property of the two bodies, by means of which a force is exerted between them at their point of contact to prevent one body from sliding on the other, is called friction the force exerted is called the force of friction.

5.1.2 Limiting Equilibrium:

Frictional force is a self-adjusting force it appears only when the external forces act and vanishes when they cease to act. The frictional force is exactly equal to the force that we exert on the body in pulling it and if we increase the pull, the frictional force also increases.

It must be noted that the frictional force between any two rough surfaces cannot go on increasing to balance the external forces. Ultimately a stage is soon reached when the frictional force available is no longer sufficient to overcome the effect of the external forces and the equilibrium is about to be disturbed. The body is the just on the point of motion.

The body is then said to be in limiting equilibrium.

5.1.3 Limiting Friction:

While the body is in limiting equilibrium the frictional force also reaches the maximum or limiting value that is, it is the frictional force at the time when the body is about to move the frictional force is called limiting friction.

5.2 Coefficient of Friction:

When the body is in limiting equilibrium the magnitude of the limiting friction bears a constant ratio to the normal reaction between the two bodies in contact. This ratio depends only on the materials of which the surfaces in contact are made and not on their shapes or areas.

This constant ratio is called the coefficient of friction. We denote it by μ.

Thus if $\vec{F}$ denotes the frictional force and $\vec{R}$ the normal reaction then when the equilibrium is limiting.

$$\frac{F}{R} = \mu \text{ or } F = \mu R$$

(ie) the limiting friction is $\mu\vec{R}$

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5.3 Laws of Friction:

Friction is not a mathematical concept. It is a physical reality. The results of physical observation and experiment are formulated as the laws of Friction.

Law: 1

When two bodies are in contact, the direction of friction on one of them at the point of contact is opposite to the direction in which the point of contat would commence to move.

Law: 2

When there is equilibrium, the magnitude of friction is just sufficient to prevent the body from moving.

Law: 3

The magnitude of the limiting friction always bears a constant ratio to the normal reaction and this ratio depends only on the substances of which the bodies are composed.

Law: 4

The limiting friction is independent of the extent and shape of the surface is contact, so long as the normal reaction is unaltered.

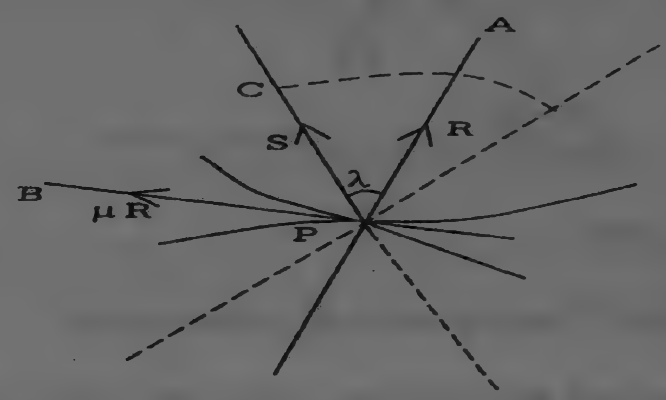
Law: 5(Law of dynamical Friction)

When motion ensues by one body sliding over the other the direction of friction is opposite to that of motion; the magnitude of the friction is independent of the velocity of the point of contact but the ratio of the friction to the normal reaction is slightly less when the body moves, than when it is in limiting equilibrium.

5.4 Angle of Friction:-

When two rough bodies in contact are in limiting equilibrium, limiting friction is called into play, The angle which the resultant of this limiting

frictional force $\mu\vec{R}$ and the normal reaction $\vec{R}$ makes with the normal to the surface in contact (PA in the diagram) is called 'angel of friction'. It is denoted by λ.



Note that R and μ R are the resolved parts of $\vec{S}$ along PA and PB respectively and so $R = S \text{ Cos } \lambda$ and $\mu R = \text{Sin}\lambda$, giving $\mu = \tan\lambda$.

Thus the coefficient of friction is equal to the tangent of the angel of friction.

5.5 Cone of Friction:

For two given bodies, the limiting frictional force being $\mu\vec{R}$, the maximum inclination of the resultant reaction $\vec{S}$ to the common normal at the point of contact of the surfaces is $\lambda = \tan^{-1} \mu$ with the normal PA as axis and the point of contact P as the vertex if we describe a cone with semi-vertical angel λ, the resultant reaction through p will always have a direction lying within the cone and in the limiting case on the surface of it. Such a cone is called the cone of friction.

5.5.1 Equilibrium of a particle on a rough inclined plane:

Let a particle of weight W be placed at A on a rough inclined plane, whose inclination to the horizontal is θ.

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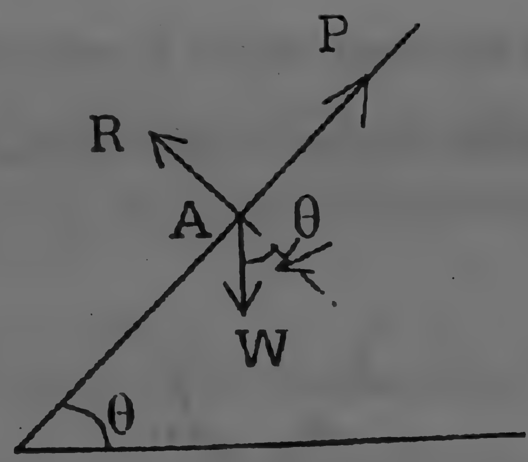


The forces acting on it are:

i) its weight W acting vertically downwards

ii) the frictional force F acting along the inclined plane upwards. (If there had been no friction, the body would have a tendency to move downwards. Hence friction will act upwards.)

iii) The normal reaction R, perpendicular to the plane.

Resolving along and $\perp^r$ to the plane, we get

$F = W \sin\theta$ (1)

and $R = W \cos\theta$ (2)

$\frac{F}{R} = \tan\theta$ (3)

We know that $\frac{F}{R}$ is always $< \mu$.

Hence for equilibrium $\tan\theta < \mu$

(ie) $\tan\theta < \tan\lambda$,

λ being the angle of friction or $\theta < \mu$.

Suppose θ, the inclination of the plane is gradually increased.

When $\theta = \lambda$, then $\frac{F}{R} = \tan\lambda = \mu$.

In this case, the equilibrium becomes limiting and the particle is just on the point of sliding down.

Hence we have the following theorem:

If a body be placed on a rough inclined plane and be on the point of sliding down the plane under the action of its weight and the reaction of the plane only, the angle of inclination of the plane to the horizon is equal to the angle of friction.

The inclination ($\lambda = \tan^{-1}\mu$) of the inclined plane when the body just begins to slip is called the angle of repose. Hence the above theorem is stated as:

The angle of repose of a rough inclined plane is equal to the angle of friction.

Note:

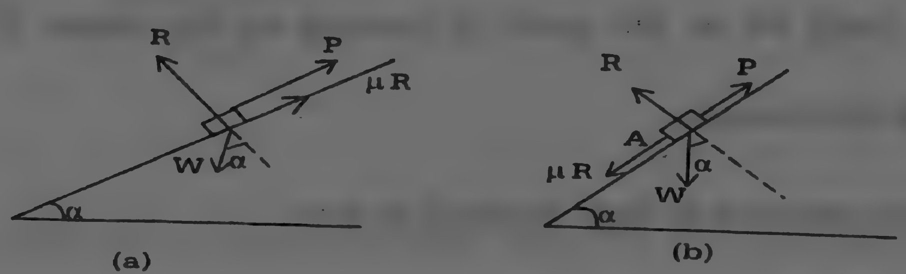
If should be noted that the angle of repose of a rough inclined plane is equal to the angle of friction, only when there are no external force acting on the body.

5.5.2 Equilibrium of a body on a rough inclined plane under a force parallel to the plane:

Theorem:

A body is at rest on a rough plane inclined to the horizon at an angle greater than the angle of friction and is acted upon by a force, parallel to the plane and along the line of greatest slope, to find the limits between which the force must lie.

Proof:



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Let α be the inclination of the plane to the horizon, W the weight of the body and R the normal reaction.

Case .1 Refer to fig. (a)

Let the body be on the point of moving down the plane. Then limiting friction acts up the plane and = μR.

Let P be the force required to keep the body at rest.

Resolving along and perpendicular to the plane, we have

$$P + \mu R = W \sin\alpha \qquad (1)$$

$$\text{and } R = W \cos\alpha \qquad (2)$$

Substituting for R from (2) in (1), we get

$$P = W \sin\alpha - \mu W \cos\alpha$$

If λ is the angle of friction,

We know that $\mu = \tan\lambda$.

$$\therefore P = W(\sin\alpha - \tan\lambda \cos\alpha)$$

$$= W\frac{(\sin\alpha\cos\lambda - \sin\lambda\cos\alpha)}{\cos\lambda}$$

$$= \frac{W\sin(\alpha-\lambda)}{\cos\lambda}$$

Let this value of P be P_1.

$$\text{Then } P_1 = \frac{W\sin(\alpha-\lambda)}{\cos\lambda} \qquad (3)$$

Since $\alpha > \lambda$, P_1 is positive

Case. 2 As in Fig (b).

Let the body be on the point of moving up the plane. Then limiting friction μR acts downwards.

Let P be the force required to keep the body at rest.

Resolving as before.

We get

$$P - \mu R = W \sin\alpha \qquad (4)$$

$$\text{and } R = W \cos\alpha \qquad (5)$$

Hence

$$P = \mu W \cos\alpha + W \sin\alpha$$

$$= W(\tan\lambda \cos\alpha + \sin\alpha)$$

$$= \frac{W(\sin\lambda\cos\alpha + \cos\lambda\sin\alpha)}{\cos\lambda}$$

$$= \frac{W\sin(\alpha+\lambda)}{\cos\lambda} = P_2 \text{ (Say)}$$

$$P_2 = \frac{W\sin(\alpha+\lambda)}{\cos\lambda} \qquad (6)$$

Now if P is $> P_2$, the body will move up the plane.

$\therefore$ P_2 is the limiting value of P, which is necessary to keep the body in equilibrium, without moving upwards.

If P is $< P_1$, the body will move down the plane.

$\therefore$ P_1 is the limiting value of P, which is necessary to keep the body in equilibrium, without moving downwards.

Hence, if P lies between P_1 and P_2, the body will remain in equilibrium and is not in the point of motion is either direction.

Hence, for equilibrium, the force P must lie between the values

$$\frac{W\sin(\alpha-\lambda)}{\cos\lambda} \text{ and } \frac{W\sin(\alpha+\lambda)}{\cos\lambda}$$

Note:

The value of P_2 may be obtained form that of P_1 by changing the sign of μ.

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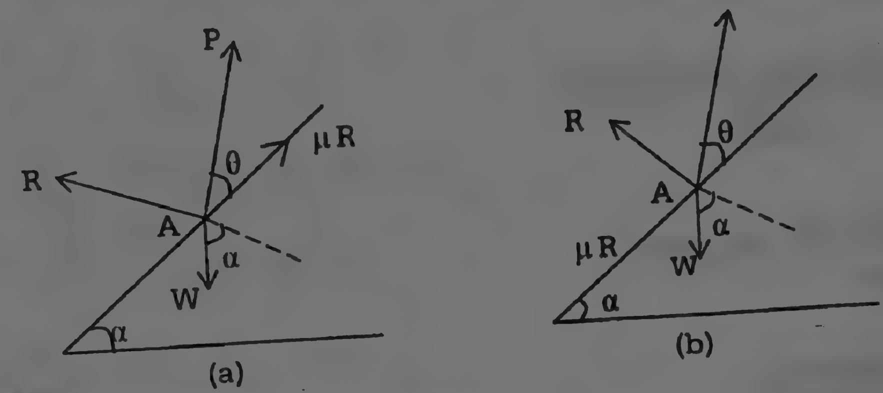
5.6 Equilibrium of a body on a rough inclined plane under any force:

Theorem:

A body is at rest on a rough inclined plane of inclination α to the horizon being acted on by a force making an angle θ with the plane to find the limits between which the force must lie and also to find the magnitude and direction of the least force required to drag the body up the inclined plane.

Proof:



Let W be the weight of the body, P the force acting at an angle θ with the plane and R the normal reaction.

Case 1:

In fig (a) the body in just on the point of moving down the plane. Then limiting friction μ R acts upwards.

Resolving the force along and $\perp^r$ to the plane,

We get, $P\cos\theta + \mu R = W\sin\alpha$ (1)

and $P\sin\theta + R = W\cos\alpha$ (2)

Substituting the value of R from (2) in (1)

We get

$$P\cos\theta + \mu(W\cos\alpha - P\sin\theta) = W\sin\alpha$$

(ie) $P(\cos\theta - \mu\sin\theta) = (W\sin\alpha - \mu\cos\alpha)$

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$$P = \frac{W(Sin\alpha - \mu Cos\alpha)}{Cos\theta - \mu Sin\theta}$$

If λ is the angle of friction,

We know the $\mu = \tan\lambda$

$$\therefore\ P = \frac{W(Sin\alpha - \tan\lambda Cos\alpha)}{Cos\theta - \tan\lambda Sin\theta}$$

$$= \frac{W(Sin\alpha Cos\lambda - Sin\lambda Cos\alpha)}{Cos\theta Cos\lambda - Sin\lambda Sin\theta}$$

$$P = \frac{WSin(\alpha - \lambda)}{Cos(\theta + \lambda)}$$

Let this value of be P_1

$$P_1 = \frac{WSin(\alpha - \lambda)}{Cos(\theta + \lambda)} \qquad (3)$$

Case. 2

In fig. (b), the body in just on the pint of moving up the plane. Then limiting friction μR acts downwards,

Resolving the forces as before.

$$P\ Cos\ \theta - \mu R = WSin\alpha \qquad (4)$$

$$P\ Sin\ \theta + R = WCos\alpha \qquad (5)$$

Substituting the value of R from (5) in (4),

We get

$$PCos\theta - -\mu(WCos\alpha - PSin\theta) = WSin\alpha$$

(ie) $\quad P\ (Cos\theta + \mu Sin\theta) = W(Sin\alpha + \mu Cos\alpha)$

$$P = \frac{W(Sin\alpha + \mu Cos\alpha)}{Cos\theta + \mu Sin\theta}$$

$$= \frac{W(Sin\alpha + \tan\lambda Cos\alpha)}{Cos\theta + \tan\lambda Sin\theta}$$

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$$= \frac{W(Sin\alpha Cos\lambda + Sin\lambda Cos\alpha)}{Cos\theta Cos\lambda + Sin\lambda Sin\theta}$$

$$P = \frac{WSin(\alpha+\lambda)}{Cos(\theta-\lambda)}$$

Let this value of P be P_2.

$$P_2 = \frac{WSin(\alpha-\lambda)}{Cos(\theta-\lambda)} \tag{6}$$

P_1 and P_2 are the limiting values of the force P, necessary to keep the body in equilibrium.

Hence if lies between P_1 and P_2, the body will remain in equilibrium.

Note:

(1) the value of P_2 may be obtained form that of P_1 by changing the sign of μ.

(2) By putting $\theta = 0$, the discussion of above theorem, gives the results of discussion of 5.5.2.

Corollary:

We can find the direction and magnitude of the least force required to drag the body up the inclined plane.

From Case 2,

$$P = \frac{WSin(\alpha+\lambda)}{Cos(\theta-\lambda)}$$

Since α, and W and λ are constants, P is least if cos $(\theta-\lambda)$ is greates.

(ie) if Cos $(\theta-\lambda)$ is greatest.

(ie) if Cos $(\theta-\lambda) = 1$.

This happens when $(\theta-\lambda) = 0$.

(ie) when $\theta = \lambda$.

In that case, value of

$P = W \sin(\alpha + \lambda)$

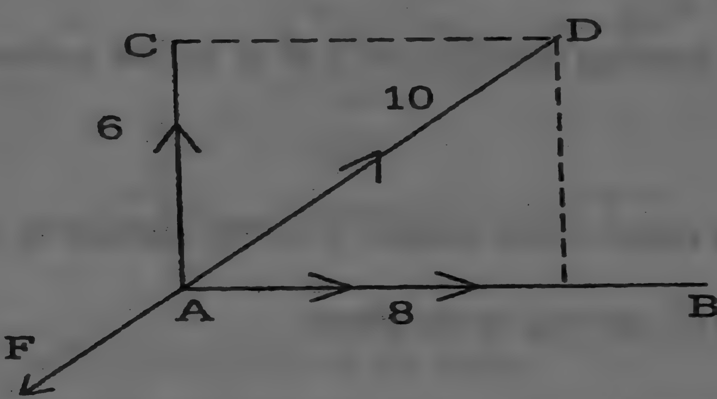
Hence the force required to move the body up the plane will be least when it is applied in a direction marking with the inclined plane an angle equal to the angle of friction. This result is sometimes stated as:

" The best angle of friction up a rough inclined plane is the angle of friction".

Example: 1

A particle of weight 30kgs. resting on a rough horizontal plane is just on the point of motion when acted on by horizontal forces of 6kg wt and 8kg wt. at right angles to each other. Find the coefficient of friction between the particle and the plane and the direction in which the friction acts.

Solution:



Let AB = 8, AC = 6 represent the directions of the forces. A being the particle.

The resultant force = $\sqrt{8^2+6^2} = \sqrt{100}$ = 10 kg. Weight and this acts along AD. Making an angle $\cos^{-1}\left(\frac{4}{5}\right)$ with the 8kg. force.

The particle tends to move in the direction AD of the resultant force and hence friction acts in the opposite direction DA.

Let F be the frictional force.

As motion just begins, magnitude of F is equal to that of the resultant

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force.

$$F = 10 \quad (1)$$

If R is the normal reaction on the particle, $R = 30$ (2)

If μ is the coefficient of friction as the equilibrium is limiting, $F = \mu R$.

$$\text{(ie) } 10 = \mu 30 \text{(or)} \mu = \frac{10}{30} = \frac{1}{3}.$$

Example: 2

A heavy particle rests in limiting equilibrium on a rough inclined plane. It is just on the point of moving up the plane along the line of greatest slope when a force equal to the weight its applied in that direction. Find the coefficient of friction.

Solution:

Let W be the weight of the particle resting at a point O on the line of greatest slope PQ is limiting equilibrium. Let α be the inclination of the plane to the horizon.

When a force equal to the weight W of the particle is applied to it along OQ, it is on the point of moving up the plane.

If $\vec{R}$ is the normal reaction at O, the limiting friction $\mu\vec{R}$ acts down the plane along OP.

Resolving the forces along and $\perp^r$ to the plane.

We get,

$$\mu R + W\text{Sin}\alpha = W$$

$$R = W\text{ Cos }\alpha$$

Hence $\mu(W\text{Cos}\alpha) + W\text{Sin}\alpha = W$

giving $\mu = \dfrac{1 - \text{Sin}\alpha}{\text{Cos}\alpha}$

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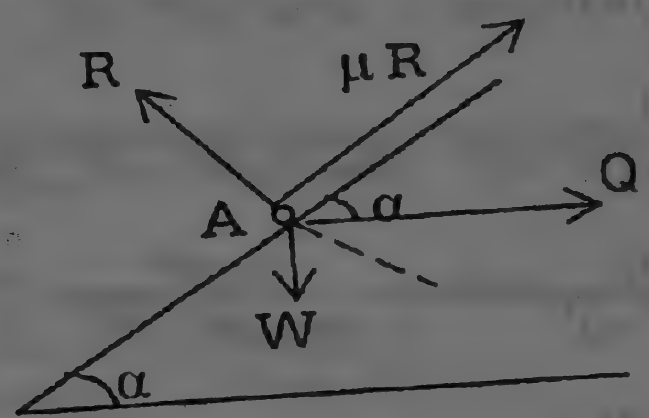
Example: 3

A weight can be supported on a rough inclined plane by a force P acting along the plane or by a force Q acting horizontally. Show that the weight is $\frac{PQ}{\sqrt{(Q^2 Sec^2\lambda - P^2)}}$ where λ is the angle of friction.

Solution:

Let W be the weight and α the angle of inclination of the plane. R is the normal reaction. When the weight is just on the point of moving down, limiting friction. μR acts upwards. A horizontal force Q keeps the weight in equilibrium.



Resolving along and perpendicular to the plane.

$$\mu R + QCos\ \alpha = WSin\alpha$$

(1)

$$R = WCos\alpha + QSin\alpha$$

(2)

Putting the value of R from (2) in (1)

We have

$$\mu(W\cos\alpha + QSin\alpha) + QCos\alpha = WSin\alpha$$

(ie) $Q\ (\mu Sin\alpha + \alpha) = W\ (Sin\alpha - \mu Cos\alpha)$

(3)

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The same weight W is supported by a force P acting along the plane.

From (2) by the theorem 5.5.2,

We have

$$P = \frac{W\operatorname{Sin}(\alpha-\lambda)}{\operatorname{Cos}\lambda} = \frac{W}{\operatorname{Cos}\lambda}(\operatorname{Sin}\alpha\operatorname{Cos}\lambda - \operatorname{Cos}\alpha\operatorname{Sin}\lambda) \qquad (4)$$

From (3) $\operatorname{Cos}\alpha(Q+\mu W) = \operatorname{Sin}\alpha(W-\mu Q)$

$$\frac{\operatorname{Cos}\alpha}{W-\mu Q} = \frac{\operatorname{Sin}\alpha}{Q+\mu W}$$

and each $= \dfrac{\sqrt{\operatorname{Cos}^2\alpha + \operatorname{Sin}^2\alpha}}{\sqrt{(W-\mu Q)^2 + (Q+\mu W)^2}}$

$$= \frac{1}{\sqrt{W(1+\mu^2) + Q^2(1+\mu^2)}} = \frac{1}{\sqrt{1+\mu^2}.\sqrt{W^2+Q^2}}$$

$$= \frac{1}{\sqrt{1+\tan^2\lambda}\sqrt{W^2+Q^2}} = \frac{1}{\sqrt{\operatorname{Sec}^2\lambda}.\sqrt{W^2+Q^2}} = \frac{1}{\operatorname{Sec}\lambda\sqrt{W^2+Q^2}}$$

$$\therefore \operatorname{Cos}\alpha = \frac{W-\mu Q}{\operatorname{Sec}\lambda\sqrt{W^2+Q^2}} \text{ and} \operatorname{Sin}\alpha = \frac{Q+\mu W}{\operatorname{Sec}\lambda\sqrt{W^2+Q^2}}$$

Substituting these in (4)

We have

$$P = \frac{W}{\operatorname{Cos}\lambda}\left(\frac{(Q+\mu W)\operatorname{Cos}\lambda - (W-\mu Q)\operatorname{Sin}\lambda}{\operatorname{Sec}\lambda\sqrt{w^2+Q^2}}\right)$$

$$= \frac{W}{\sqrt{w^2+Q^2}}[Q(\operatorname{Cos}\lambda + \mu\operatorname{Sin}\lambda) + W(\mu\operatorname{Cos}\lambda - \operatorname{Sin}\lambda)]$$

$$= \frac{W}{\sqrt{w^2+Q^2}}[Q(\operatorname{Cos}\lambda + \frac{\operatorname{Sin}\lambda}{\operatorname{Cos}\lambda}\operatorname{Sin}\lambda) + W(\frac{\operatorname{Sin}\lambda}{\operatorname{Cos}\lambda}\operatorname{Cos}\lambda - \operatorname{Sin}\lambda)]$$

$$= \frac{W}{\sqrt{w^2+Q^2}}\frac{Q(\operatorname{Cos}^2\lambda + \operatorname{Sin}^2\lambda)}{\operatorname{Cos}\lambda} = \frac{WQ\operatorname{Sec}\lambda}{\sqrt{W^2+Q^2}}$$

$$P^2 = \frac{W^2Q^2\operatorname{Sec}\lambda}{W^2+Q^2} \Rightarrow P^2(W^2+Q^2) = W^2Q^2\operatorname{Sec}^2\lambda^2$$

(or) $W^2\ (Q^2\sec^2\lambda - P^2) = P^2Q^2$

$$W^2 = \frac{P^2Q^2}{Q^2\operatorname{Sec}^2\lambda - P^2} \text{ or } W = \frac{PQ}{\sqrt{Q^2\operatorname{Sec}^2\lambda - P^2}}.$$

Example: 4

Show that the horizontal force, which can support a heavy mass on a smooth inclined plane of inclination α to the horizon can just sustain it in limiting equilibrium on a rough plane inclined to the horizon at an angle of $(\alpha+\lambda)$ where λ is the angle of friction.

Solution:

Referring to equation (3) of the previous worked example:

We have

$$Q = \frac{W(\operatorname{Sin}\alpha - \mu\operatorname{Cos}\alpha)}{\mu\operatorname{Sin}\alpha + \operatorname{Sec}\alpha} \qquad (1)$$

$$= \frac{W(\operatorname{Sin}\alpha - \frac{\operatorname{Sin}\lambda}{\operatorname{Cos}\lambda}\operatorname{Cos}\alpha)}{\frac{\operatorname{Sin}\lambda}{\operatorname{Cos}\lambda}\operatorname{Sin}\alpha + \operatorname{Cos}\alpha} = \frac{W\operatorname{Sin}(\alpha-\lambda)}{\operatorname{Cos}(\alpha-\lambda)} \qquad (2)$$

This gives the horizontal force Q which just keeps the weight W in limiting equilibrium on a rough plane of inclination α.

Let Q_1 be the horizontal force which keeps W is limiting equilibrium on a rough plane of inclination $\alpha+\lambda$.

Q_1 can be got from Q by changing α in to $\alpha+\lambda$.

$$Q_1 = \frac{W\operatorname{Sin}(\alpha+\lambda-\lambda)}{\operatorname{Cos}(\alpha+\lambda-\lambda)} = W\tan\alpha \qquad (3)$$

Let Q_2 be the horizontal force which keeps w in equilibrium on a smooth plane of inclination α.Q_2 can be got from the value of Q gives in (1) by putting μ=0.

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$$Q_2 = \frac{W\sin\alpha}{\text{Cos}\alpha} = W\tan\alpha \qquad (4)$$

From (3) and (4)

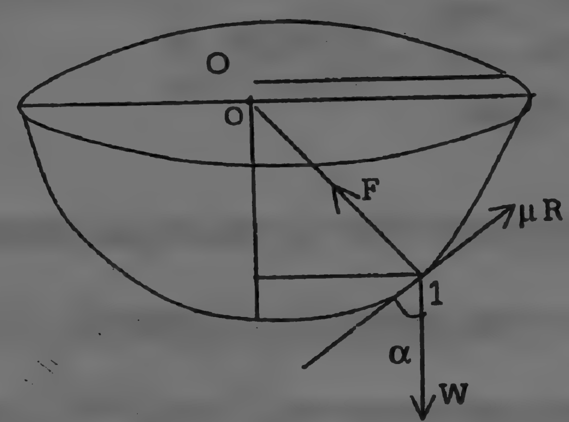
We find that $Q_1 = Q_2$

Example: 5

An insect tries to crawl up the inside of fixed hemispherical bowl of radius R. How high can it travel? If the coefficient of friction between its feet and the bowl in $\frac{1}{3}$?

Solution:

Let A be lowest point of the fixed hemispherical bowl of radius R. Let the insect of weight $\vec{W}$ (Say), Crawl upto 1. When it is about to slip back. Then the maximum friction $\mu\vec{F}$ is called into play $\vec{F}$ being the normal reaction through O, the centre of the bowl.



Since the three forces $\vec{F}, \mu\vec{F}, \vec{W}$, meeting at 1 are in limiting equilibrium by Lami's theorem, we get

$$\frac{F}{\text{Sin}(180^\circ - \alpha)} = \frac{\mu F}{\text{Sin}(90^\circ + \alpha)} = \frac{W}{\text{Sin}90^\circ}$$

Where α is the angle that $\mu\vec{F}$ makes with the vertical

Hence $F = W\ \text{Sin}\ \alpha$

And $\mu F = W\text{Cos}\alpha$

So that $\mu = \text{Cot}\alpha$

Since $\mu = \frac{1}{3}$

$\text{Cot}\ \alpha = \frac{1}{3}$

The height of the point 1 about A = AP = OA – OP = R – R Sin α

as $\angle$OIP= α

$= R\,(1-\text{Sin}\alpha) = R\ (1-\frac{3}{\sqrt{10}})$ as $\text{Cot}\ \alpha = \frac{1}{3}$

$$\because \text{Cot}\ \alpha = \frac{1}{3} \Rightarrow \frac{\text{Cos}\alpha}{\text{Sin}\alpha} = \frac{1}{3}$$

$$\Rightarrow 3\text{Cos}\alpha = \text{Sin}\alpha$$

$$\Rightarrow 9\text{Cos}^2\alpha = \text{Sin}^2\alpha$$

$$\Rightarrow 9(1-\text{Sin}^2\alpha) = \text{Sin}^2\alpha$$

$$\Rightarrow 9 = 10\text{Sin}^2\alpha$$

$$\Rightarrow \text{Sin}^2\alpha = \frac{9}{10}$$

$$\Rightarrow \text{Sin}\alpha = \frac{3}{\sqrt{10}}$$

Exercises:

1. A body of weight 4kgs. rests in limiting equilibrium on an inclined plane whose inclination is 30^0. Find the coefficient of friction and the normal reaction.
2. A body of weight 4kgs. rests in limiting equilibrium on a rough plane whose slope is 30^0, if the plane is raised to a slope of 60^0, find the force along the plane required to support the body.

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3. If the force, which acting parallel to a rough plane of inclination α to the horizon is just sufficient to draw a weight up, be n times the force which will just let it be on the point of slipping down, show that

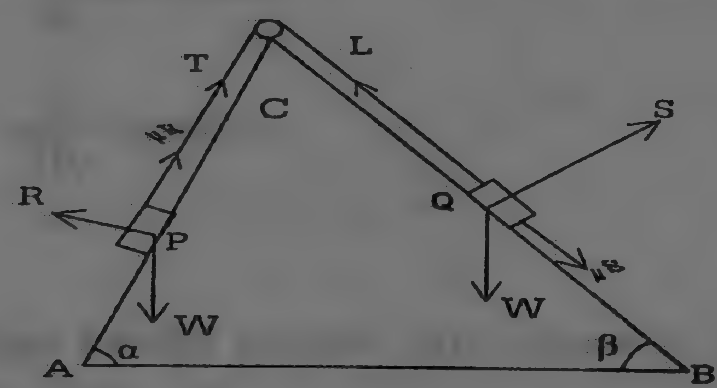
$$\tan\alpha = \mu\frac{n+1}{n-1}$$

Example: 6

Two particles P and Q each of weight W on two equally rough inclined planes CA and CB of the same height, placed back to back are connected by a light string which passes over the smooth top edge C of the planes. Show that if the particles are on the point of slipping the difference of the inclinations of the planes is double the angle of friction.

Solution:

Let α and β be the inclinations of the planes CA and CB; R, S be the normal reactions of the planes, T the tensions of the planes, T the tension of the string and μ the coefficient of friction (which is the same for both planes).



Let P be on the point of moving downwards. Then Q will be moving upwards.

Limiting friction μR will act on P upwards the inclined plane and limiting friction μs will act on Q downwards the inclined plane.

Considering the equilibrium of P and resolving along and $\perp^r$ to the plane CA.

We have

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$\mu R + T = W\sin\alpha$

(1)

$R = W\cos\alpha$ (2)

Substituting the value of R from (2) in (1)

$T = W\sin\alpha - \mu W\cos\alpha$ (3)

Resolving along and perpendicular to the plane CB.

We have $T = W\sin\beta + \mu s$ (4)

and $S = W\cos\beta$ (5)

Substituting (5) in (4)

We get $T = W\sin\beta + \mu W\cos\beta$ (6)

Equating the two values of T in (6) and (3)

$W(\sin\beta + \mu\cos\beta) = W(\sin\alpha - \mu\cos\alpha)$

(ie) $\mu(\cos\alpha + \cos\beta) = \sin\alpha - \sin\beta$

$$\mu = \frac{\sin\alpha - \sin\beta}{\cos\alpha + \cos\beta} = \frac{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}$$

$= \tan\dfrac{\alpha-\beta}{2}$ (7)

If λ is the angle of friction,

We know that

$\mu = \tan\lambda$

From (7) & (8)

We get $\lambda = \dfrac{\alpha-\beta}{2}$

(or) $2\lambda = \alpha - \beta$

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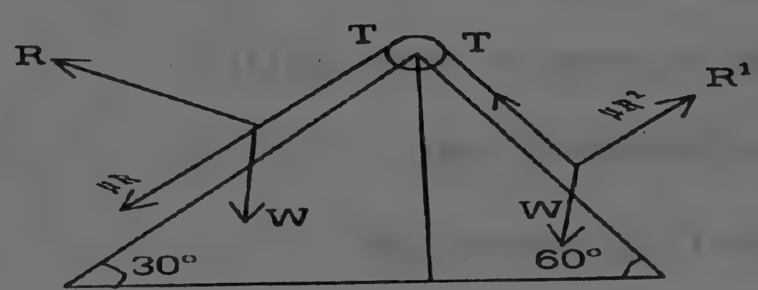
Example: 7

Two equal weights are attached to two ends of a string which pass over a smooth pulley on the top of two equally rough planes having the same altitude and placed back to back the inclination of the planes to the horizontal being 30^o and 60^o respectively. If the system be in limiting equilibrium. Show that the coefficient of friction in 2- $\sqrt{3}$.

Solution:

Since the inclination of the second plane. (namely, 60^o) is greater, the weight W on this plane is on the point of moving down the plane and so the weight W on the other plane is on the point of moving up.



The pulley being smooth, the tension is the two parts of the string are the same, say $\vec{T}$.

The normal reaction of the planes are $\vec{R}$ and $\vec{R'}$ and $\mu\vec{R}$ acts down the plane (of inclination 30^o) while $\mu\vec{R'}$ acts up the other plane.

Resolving the forces normal to the planes separately,

We get

$$R = W \cos 30^o = \frac{\sqrt{3}}{2} W$$

and $R' = W \cos 60^o = \dfrac{1}{2} W$

Resolving the forces on each planes along the plane,

We get

$$T - \mu.R = WSin30^o$$

$$\text{(or) } T = \mu.\frac{\sqrt{3}}{2} + W.\frac{1}{2}$$

$$\text{and } T + \mu R^1 = W \text{ Sin } 60^0 \Rightarrow T = W.\ \frac{\sqrt{3}}{2} - \mu.\frac{1}{2}W$$

These give $\mu\ \frac{\sqrt{3}}{2}W + W.\frac{1}{2} = W.\frac{\sqrt{3}}{2} - \mu.\frac{1}{2}W$

$$\text{(or) } \mu(\sqrt{3}+1) = \sqrt{3}-1$$

Hence $\mu = \frac{\sqrt{3}-1}{\sqrt{3}+1} x \frac{\sqrt{3}-1}{\sqrt{3}-1}$

$$= \frac{(\sqrt{3}-1)^2}{3-1} = \frac{3+1-2\sqrt{3}}{2}$$

$$= \frac{4-2\sqrt{3}}{2}$$

$$= \frac{2(2-\sqrt{3})}{2}$$

$$\mu = 2-\sqrt{3}$$

Example: 8

A Straight rod rests with one end A on a rough horizontal plane and the other end B against a smooth vertical wall. If ℓ be the length of rod and a, the distance of its gravity from show that the centre of the inclination of the rod to the wall when on the point of slipping is $\tan^{-1}\left(\frac{\ell\mu}{a}\right)$ where μ is the coefficient of friction.

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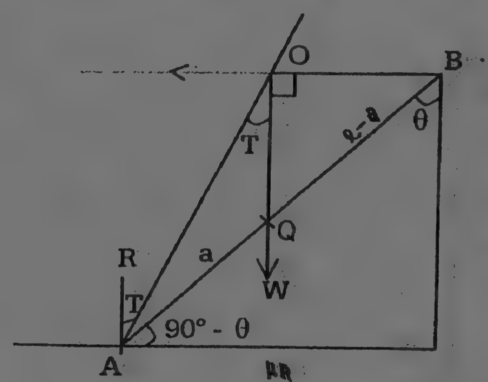
Solution:

Since the wall is smooth, the reaction $\vec{S}$ at B is perpendicular to the wall. The rod being on the point of slipping the reaction at A on the ground makes an angle λ (angle of friction) with the vertical through A.

If G is C. G of rod AB the line of action of the weight $\vec{W}$ through G passes through O, the point where the reactions meet.



(ie) OG is Vertical

In $\Delta AOB, \angle AOG = \lambda,$

$\angle GOB = 90^0$

$AG = a$ and $GB = \ell - a$

Applying the trigonometrically result to ΔAOB

We get

$$(AG + GB) \text{ Cot} \angle OGB = AG \text{ Cot} \angle AOG - GB \text{ Cot} \angle GOB$$

(ie) $\ell \cot\theta = a \text{ Cot } \lambda$

(or) $\tan\theta = \dfrac{\ell}{a}\tan\lambda$

$= \dfrac{\ell\mu}{a}$

$\therefore \theta = \tan^{-1}\left(\dfrac{\mu}{a}\right)$

5.7 Problems on Friction:

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Problems on the equilibrium of a rigid body or bodies involving friction are of varied types. They can be classified as follows.

a) These in which the equilibrium of the rigid body can be broken only by sliding, the direction of motion being obvious.

For example,

Consider a ladder resting on rough ground against a rough wall in a vertical plane $\perp^r$ to the wall. If no other forces act, it is clear that the lower end of the ladder slips away from the wall and the upper end slips downwards at the same time. One end cannot slip without the other. Other examples of this type are, a rod resting over two rough pegs, a rod resting within a rough hollow sphere etc.

b) Those in which equilibrium is broken either by sliding or tilting. For example, a black of cylinder resting on a rough plane which is gradually tiled. If the vertical through the centre of gravity comes outside the base before sliding commences, the body will topple over before sliding.

c) Those in which we have a non – rigid body such as two jointed rods AB, BC freely jointed at B and with the ends A and C resting on a rough horizontal plane. In this case, it is not necessary for slipping to occur at both A and C. It may happen that one of the rods may slip before the other does.

The method of dealing with such problems is illustrated in the following examples.

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Example: 1

A uniform ladder is in equilibrium with one end resting on the ground and the other against a vertical wall: If the ground and wall be both rough, the coefficients of friction being μ and μ' respectively, and if the ladder be on the point of slipping at both ends, show that θ, the inclination of the ladder to horizon is given by

$$\tan\theta = \frac{1-\mu\mu^{1}}{2\mu}$$

Find also the reactions at the wall and ground.

Solution:

First method: (Refer to 'a')

Let AB (=2a) be the ladder G its centre of gravity and W its weight. Let R and S be the normal and S be the normal reactions acting on the ladder at the ground and wall.

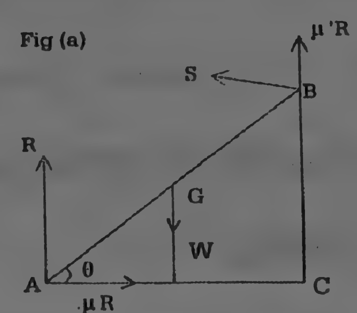

Fig.(a)

The lower end A will have a tendency to move away from the wall and hence friction there will be acting in the direction AC. The upper end B will have a tendency to move downwards and hence friction there will be acting upwards. Since both ends are slipping the frictions at A and B are limiting. They are μR and $\mu' S$ as marked in the figure.

Resolving horizontally, $S = \mu R$ (1)

Resolving Vertically, $\mu^1 S + R = W$ (2)

Taking moments about A,

$$S.BC + \mu' S.AC = W.\ AE.$$

(ie) $S.2a \sin\theta + \mu^1 2 \cos a \cos\theta = w.a \cos\theta$ (3)

Putting the value of S from (1) in (2)

We have

$$\mu'\mu R + R = W \Rightarrow R[\mu'\mu + 1] = W$$

(ie) $R = \dfrac{W}{1+\mu\mu'}$ (4)

Then from (1) $S = \dfrac{\mu W}{1+\mu\mu'}$ (5)

(4) and (5) give the reactions.

Putting the value of S from (5) in (3)

We get $\dfrac{\mu W}{1+\mu\mu'} 2\sin\theta + \dfrac{\mu'\mu W}{1+\mu\mu'} 2\cos\theta = W\cos\theta$

(ie) $2\ \mu\sin\theta + 2\mu\mu'\cos\theta = (1+\mu\mu')\cos\theta$

(ie) $2\ \mu\sin\theta \qquad = (1-\mu\mu')\cos\theta$

(or) $\dfrac{\sin\theta}{\cos\theta} = \dfrac{1-\mu\mu'}{2\mu} = \tan\theta$ (6)

<u>Second Method: - (Refer to fig (b)</u>

The two forces, friction and normal reaction at A can be compounded into a single force "Similarly, the two forces, friction and normal reaction at B can be compounded into a single force. Let these two resultant reactions meet at L. For equilibrium, the third force, the weight of the ladder must pass through L.

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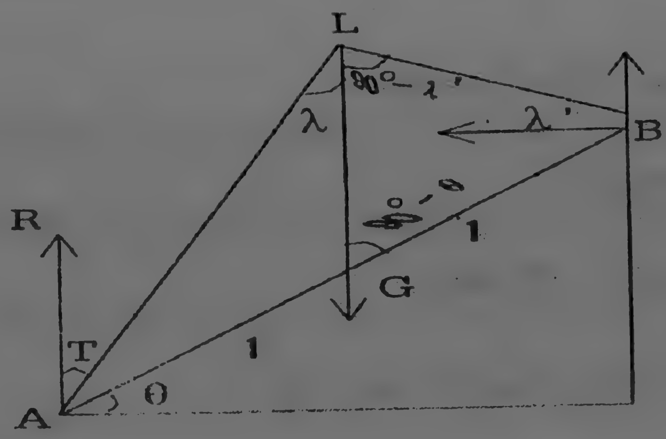


Fig (b)

Hence LG is vertical. As the equilibrium is limiting, these resultant reactions must make angles λ, λ' with the normal at A and B [Where $\tan \lambda = \mu$ and $\tan \lambda' = \mu'$], In Δ ALB, $\angle LGB = 90^0 - \theta$, $\angle ALG = \lambda$, $\angle BLG = 90 - \lambda'$ and AG: GB = 1:1

Using the first trigonometric equation

We have $(1 + 1) \text{Cot}\,(90^0 - \theta) = 1. \text{Cot}\,\lambda - 1. \text{Cot}\,(90^0 - \lambda')$

(ie) $2\tan\theta = \text{Cot}\,\lambda - \tan\lambda'$.

$$= \frac{1}{\tan\lambda} - \tan\lambda'$$

$$= \frac{1}{\mu} - \mu' = \frac{1-\mu\mu'}{\mu}$$

$$\therefore \tan\theta = \frac{1-\mu\mu'}{\mu}$$

Note:

1. By this method, we cannot find the reactions. They have to be found by resolving the forces.
2. This geometrical method is particularly useful when we require only the position of a body in limiting equilibrium as we obtain the result without introduction the reactions and having to solve the equations.

Example: 2

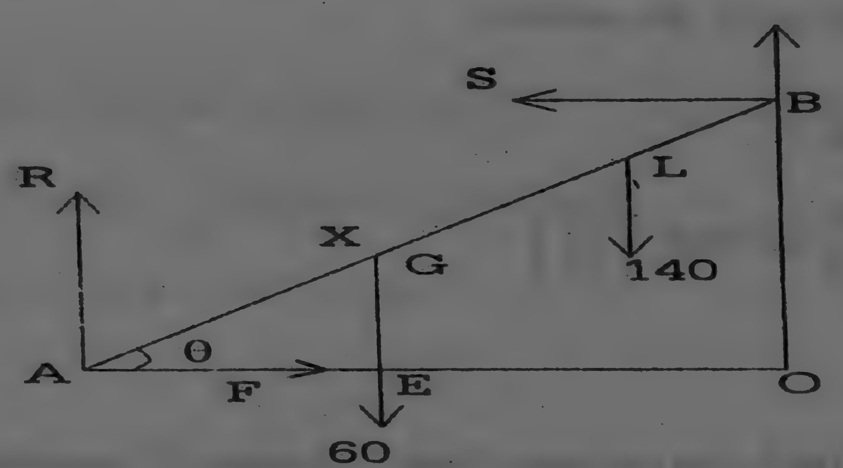
A ladder 20 meters long with its centre of gravity 8 meters up form the bottom, weights 60kgms and rests at an angle of θ to the ground against a smooth vertical wall. The coefficient of friction between the ladder and the ground is $\frac{1}{2}$, Find the least value of θ which will enable a man weighing 140 kgms. to reach the top without the ladder slipping.

Solution:

Let L be the position of the man on the ladder AB such that AL = x.

The forces acting on the ladder are:

1) Its weight 60kgs acting vertically downwards at G where AG = 8.
2) Reaction R at A, $\perp^r$ to the ground.
3) Friction F at A towards the wall as the end A will begin to slip in the direction OA.
4) Normal reaction S at B due to contact with the wall (AS the wall is smooth, there is no friction at B.)
5) The weight of the man 140kgs. at L acting vertically downwards



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Resolving horizontally and vertically,

$F = S$ (1)

$R = 140 + 60 = 200$ (2)

Taking moment about A.

$$60\ \overrightarrow{AE} + 140\ \overrightarrow{AF} = S.\overrightarrow{BC}$$

S 20 Sin θ = 60. 8 Cos θ + 140. x Cos θ (3)

$$\therefore \quad S = \frac{480 \operatorname{Cos}\theta + 140\, x \operatorname{Cos}\theta}{20 \operatorname{Sin}\theta}$$

$$= (24 + 7x) \operatorname{Cot}\theta \quad (4)$$

Now, $\frac{F}{R} = \frac{S}{200} = \frac{(24+7x)\operatorname{Cot}\theta}{200}$

Now, $\frac{F}{R} = \frac{S}{200} = \frac{(24+7x)\operatorname{Cot}\theta}{200}$

For equilibrium,

$\frac{F}{R}$ Must be $\leq \mu$, the coefficient of friction

(ie) $\frac{(24+7x)\operatorname{Cot}\theta}{200} \geq 1/2 \text{ or } \operatorname{Cot}\theta \leq \frac{100}{24+7x}$ (5)

The condition (5) must be satisfied for values of x from 0 to 20. If the man is to reach the top safely.

It is enough if the value of θ is such that, there is no slipping when the man reaches the top.

$\therefore$ Putting x = 20 in (5), the condition

$$\operatorname{Cot}\theta \leq \frac{100}{24+140}$$

(ie) Cot $\theta \leq \frac{100}{164}$.(ie) $\tan\theta^{-1}\left(\frac{41}{25}\right)$

i.e, $\tan\theta \leq \frac{41}{25}$

As θ increase, $\tan\theta$ also increases. Hence condition (6) is satisfied if

$$\theta \leq \tan^{-1}\left(\frac{41}{25}\right)$$

$\therefore$ Least value of $\theta = \tan^{-1}\left(\frac{41}{25}\right)$

Example: 3

A Uniform ladder rests in limiting equilibrium with its lower end on a rough horizontal plane and its upper end against an equally rough vertical wall. If θ be $\tan\theta = \frac{2\mu}{-\mu^2}$ Where μ is the coefficient of friction.

Solution:

Let the ladder AB rest with its end A on rough horizontal plane, its end B on the an equally rough vertical wall.

Since AB rests in limiting equilibrium, the reaction at A and B make angle λ (the angle of friction) with the perpendicular as A and B to horizontal plane and the vertical wall.

Let these forces of reaction meet at O. The line of action of the weight $\overrightarrow{W}$ of the ladder acting through G, the mid – point of AB must pass, through O. There being only three forces is equilibrium.

(ie) OG is vertical.

Since AB is inclined to the vertical at θ,

$\angle OQB = \theta$

In $\Delta OAB, \angle AOG = \lambda$ $\qquad\qquad \angle GOB = 90^0 - \lambda$

Hence applying the trigonometrical result to the ΔOAB,

We get,

$$(AG + GB)\text{ Cot }\angle OGB = AG\text{ Cot }\angle AOG - GB\text{ Cot }\angle GOB$$

(or) $$AB\text{ Cot }\theta = \frac{1}{2}AB\text{ Cot }\lambda - \frac{1}{2}AB\text{ Cot}(90^0 - \lambda)$$

$$\text{(ie) } 2\text{ Cot }\theta = \text{Cot }\lambda - \tan\lambda$$

$$= \frac{1}{\mu} - \mu = \frac{1-\mu^2}{\mu}$$

Hence $\tan\theta = \frac{2\mu}{1-\mu^2}$

Space for Hints

Example: 4

A ladder AB rests with A resting on the ground and B against a vertical wall, the coefficients of friction on the ground and wall being μ and μ' respectively. The centre of gravity of the ladder of slipping at both ends. Show that the inclination to the ground is given by $\tan\theta = \dfrac{1-n\mu\mu'}{(n+1)\mu}$

Solution:

The ladder AB of weight $\overrightarrow{W}$ is in limiting equilibrium with its end A on the ground and the other end B on a vertical wall. The reactions $\vec{R}$ and $\vec{S}$ At and B make angles λ and λ'. (the respective angles of friction) with the normal at A and B respectively.

Let $\vec{R}$ and $\vec{S}$ meet at O.

Then the line of action of the weight $\overrightarrow{W}$ of the ladder through the C.G of the ladder, namely G must passes through O.

(ie) OG is vertical.

If θ is the angle that the ladder makes with the ground then in

ΔAOB

$\angle OGB = 90^0 - \theta$

$\angle AOG = \lambda$ and $\angle GOB = 90^0 - \lambda$

Hence applying the trigonometrical result to ΔAOB.

We get,

$(n+1)\ \text{Cot}\ \angle OGB = 1\,\text{Cot}\angle AOG - n.\text{Cot}\angle GOB.$

As G divides AB in the ration 1:n.

(ie) $(n+1)\ \text{Cot}\ (90^0 - \theta) = \text{Cot}\lambda - n\cot(90-\lambda')$

(or) $(n+1)\tan\theta = \text{Cot}\ \lambda - n\tan\lambda'$

$$= \frac{1}{\mu} - n\mu' = \frac{1-n\mu\mu'}{\mu}$$

Hence $\tan\theta = \dfrac{1-n\mu'\mu}{(n+1)\mu}$

Exercise:

1. A uniform ladder rests with its lower end on a rough horizontal ground and its upper end against a rough vertical wall, the ground and the wall being equally rough and the angle of friction being λ. Show that the greatest inclination of the ladder to the vertical is 2λ.

2. A ladder AB rests with A on a rough horizontal ground and B against an equally rough vertical wall. The centre of gravity of the ladder divides AB in the radio a:b. If the ladder is on the point of slipping. Show that the inclination θ of the ladder of the ground is given by $\tan\theta = \dfrac{a-b\mu^2}{\mu(a+b)}$ where μ is the coefficient of friction.

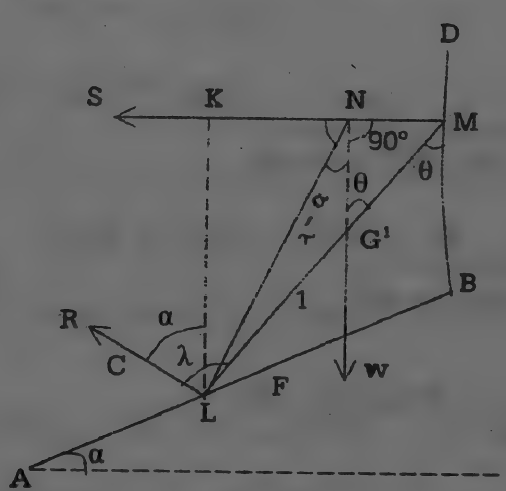
3. A uniform ladder rests with its lower end on a rough ground and its upper end against a smooth vertical wall. When in limiting equilibrium, the ladder makes with the wall an angle 30^0. Find the coefficient of friction μ.

Example: 1

A uniform rod of weight W rests with its one end against a rough inclined plane AB of inclination α and the other end against a smooth vertical wall BD, B being at a higher level them A. If θ be the inclination of the rod to the vertical in the position of limiting equilibrium. Prove that $\tan\theta = 2\tan(\lambda-\alpha)$ Where λ is the angle of friction.

Prove also that reaction of the wall is $W\tan(\lambda-\alpha)$ and the resultant reaction with ground is $W\sec(\lambda-\alpha)$.

Solution:



LM is the rod, with G its midpoint the forces acting on it are

i. its weight W acting vertically downwards at G.

ii. The reaction R at L $\perp^r$ to the inclined plane AB

iii. Limiting frictional force F at L along LB.

Space for Hints

iv. Reactions S at M normal to the wall and hence horizontal.

As the equilibrium is limiting, the two forces F and R at L can be compound into one force, the resultant reaction P making an angle λ with LC, the normai at L.

Let this resultant reaction meet the line of action of S at N.

For equilibrium, NG must be vertical.

Draw LKI|| to NG meeting the forces S at K.

$\angle$ CLK = angle between CL and LK.

= angle between their $\perp^r$, AB and the horizontal

$= \alpha$

$\angle KLN = \lambda - \alpha$

$\therefore \angle LNG = \lambda - \alpha$ and LG:GM = 1:1.

From ΔLNM, Using the first trigonometrical equation

We have

$$(1+1)\text{ Cot } \theta = 1.\text{Cot}(\lambda - \alpha) - 1.\text{Cot}\,90^0$$

$$\text{(ie) } 2 \text{ Cot } \theta = \text{Cot}((\lambda - \alpha).$$

$$\frac{2}{\tan\theta} = \frac{1}{\tan(\lambda - \alpha).}$$

(or) $\tan\theta = 2\tan(\lambda - \alpha)$

By applying Lami's theorem for the three forces P,S and W at N.

We get

$$\frac{P}{\text{Sin}\,90^0} = \frac{S}{\text{Sin}(\lambda - \alpha)} = \frac{W}{\text{Sin}(90^0 + \lambda - \alpha)}$$

$$\text{(ie) } \frac{P}{1} = \frac{S}{\text{Sin}(\lambda - \alpha)} = \frac{W}{\text{Cos}(\lambda - \alpha)}$$

$$\therefore P = \frac{W}{\text{Cos}(\lambda - \alpha)} = W\,\text{Sec}(\lambda - \alpha)$$

$$\text{and } S = \frac{W\text{Sin}(\lambda - \alpha)}{\text{Cos}(\lambda - \alpha)} = W\tan(\lambda - \alpha)$$

Example: 2

A uniform rod is in limiting equilibrium with one end resting on a rough horizontal plane and the other end on an equally rough plane inclined at an

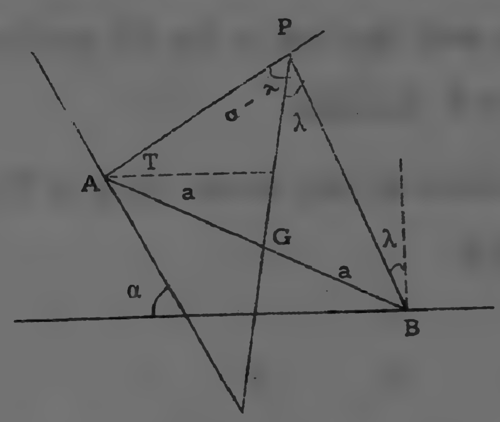
angle α to the horizon. If λ be the angle of frication and the rod be in a vertical plane, Show that if θ be the inclination of the rod with the horizon then.

$$\tan\theta = \frac{\sin(\alpha-2\lambda)}{2\sin\lambda\sin(\alpha-\lambda)}$$

Solution:

Let the rod AB of length 2a rest with its end A on the plans of inclination α with the horizon and the other end B on the horizontal plane.

Since the two planes, are equally rough, the equilibrium being limiting, the reactions at A and B make the same angle λ(the angle of friction) with the normal at A and B.



Let the reactions meet at P.

The line of action of the weight $\overrightarrow{W}$ of the rod AB acting through G, the midpoint of AB, passes through P, there being only three forces in equilibrium.

In ΔAPQ, $\angle APG$

$= 180^0 - (\angle PAQ + \angle AQP)$

$= 180^0 - (90^0 + \lambda + 90^0 - \alpha)$

$= \alpha - \lambda$

$\angle GPB \quad = \lambda \qquad$ And $\angle PGA = 90^0 - \theta$

Hence applying the trigonometrical result to ΔAPB.

We get,

$(AG + GB)\cot\angle PGA = BG\cot\angle GPB - AG\cot\angle APG$

(or) $2a\cot(90^0 - \theta) = a\cot\lambda - a\cot((\alpha-\lambda)$

$$= a\left[\frac{\cos\lambda}{\sin\lambda} - \frac{\cos(\alpha-\lambda)}{\sin(\alpha-\lambda)}\right]$$

$$\therefore \tan\theta = \frac{1}{2}\left[\frac{\sin(\alpha-\lambda)\cos\lambda - \cos(\alpha-\lambda)\sin\lambda}{\sin\lambda\sin(\alpha-\lambda)}\right]$$

Space for Hints

Space for Hints

$$\therefore \tan\theta = \frac{\sin(\alpha-2\lambda)}{2\sin\lambda\sin(\alpha-\lambda)}$$

Example: 3

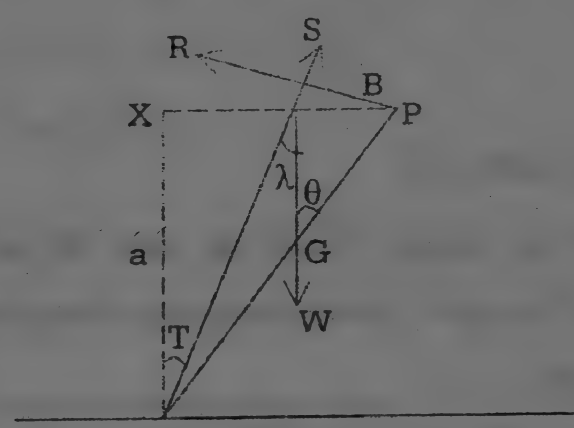
A uniform ladder of length ℓ rest on a rough horizontal ground with its upper end projecting very slightly over a horizontal smooth rail at a height a above the ground. If the ladder is about to slip down and is the coefficient of friction, show that.

$$\mu = \frac{2\sqrt{\ell^2-a^2}}{\ell^2+a^2}$$

Solution:

The ladder AB rests with the end A on the ground and its upper and B is very slightly above the rail at P, initially.

When the ladder is about to slip down. B is at P and the reaction $\vec{R}$ at B (or P) acts $\perp^r$ to the ladder AB.



Let the weight $\vec{W}$ of the ladder acting through the midpoint G meet $\vec{R}$ at O. Then the reaction $\vec{S}$ at A must pass through O there being limiting equilibrium.

The equilibrium being limiting and the ground being rough, $\vec{S}$ makes with the vertical AX the angle λ (the angle of friction). Let AB make an angle θ with AX.

Since $\angle GBO = 90^0$; $\angle GOB = 90^0 - \angle OGB = 90^0 - \theta$

Also $\angle AOG = \angle XAO = \lambda$.

Applying the trigonometrical result to the Δ AOB.

We get

$$(AG + GB)\cot\angle OGB = AG\cot\angle AOG - GB\cot\angle GOB$$

$$\ell\cot\theta = \tfrac{1}{2}\,\ell\,\cot\lambda - \tfrac{1}{2}\,\ell\cot(90^0-\theta)$$

(or) $\quad 2\,\text{Cot}\,\theta = \text{Cot}\lambda - \tan\lambda$

$$\text{Cot}\,\lambda = 2\,\text{Cot}\,\theta + \tan\,\theta = \frac{2+\tan^2\theta}{\tan\theta}$$

In $\Delta ABX, \text{Cos}\,\theta = \dfrac{a}{\ell}$ and So

$$\tan\,\theta = \sqrt{\frac{\ell^2 - a^2}{a}}$$

Hence $\mu = \tan\lambda = \dfrac{\tan\theta}{2+\tan^2\theta}$

$$= \frac{\sqrt{\ell^2-a^2}}{2} \Big/ 2 + \frac{\ell^2-a^2}{a^2}$$

$$\mu = \frac{a\sqrt{\ell^2-a^2}}{\ell^2+a^2}$$

Exercise:

1. A rod is in limiting equilibrium resting horizontally with its ends on two inclined planes at right angles, one of which makes an angle $\alpha (< 45^0)$ with the horizontal. If μ the coefficient of friction is same for both ends, prove that $\tan\mu = \dfrac{\text{Cos}\,\alpha - \text{Sin}\,\alpha}{\text{Cos}\,\alpha + \text{Sin}\,\alpha}$.

2. A uniform rod of length ℓ rests in a vertical plane against (and over) a smooth horizontal bar at a height h, the lower end of the rod being on level ground. Show that if the rod is on the point of slipping when its inclination to the horizontal is θ, then the coefficient of friction between the rod and ground is $\dfrac{\ell\,\text{Sin}\,2\theta\,\text{Sin}\,\theta}{4h - \ell\,\text{Sin}\,2\theta\,\text{Cos}\theta}$.

Example: 4

A uniform rod rests in limiting equilibrium within a rough hollow sphere. If the rod subtends an angle 2α at the centre of the sphere and if λ be the angle of friction, show that the inclination of the rod to the horizontal is $\tan^{-1}\left[\dfrac{\text{Sin}2\lambda}{\text{Cos}\,2\alpha + \text{Cos}2\lambda}\right]$

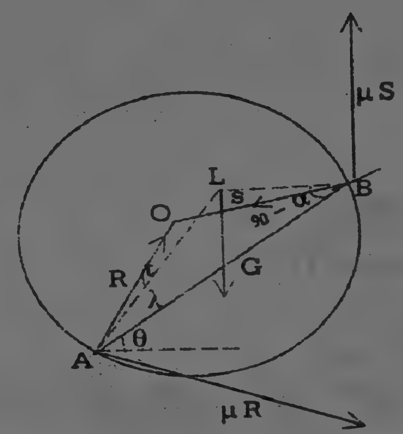
Space for Hints

Solution:

Let AB the rod, G its C.G and O is the centre of the sphere.

Since $\angle AOB = 2\alpha$

$\angle OAB = \angle OBA = 90^0 - \alpha$.



Let R and S be the normal reaction at A and B acting along AO and BO respectively.

Since B is on the point of coming down and A is on the point of going up, limiting friction μS and μR act in the directions shown in the figure.

The forces R, μR at A and S, μS at B can be compounded into single forces. The resultant reactions acting along AL and BL respectively.

Then $\angle OAL = \angle OBL = \lambda$

For equilibrium, the 3rd force.

(ie) the weight of the rod must pass through L.

(ie) LG is vertical.

Let AB make an angle θ with the horizontal.

In $\Delta \angle ALB, LAB = \angle OAB - \lambda = 90^0 - \alpha - \lambda$

$\angle LBA = \angle OBA + \lambda = 90^0 - \alpha + \lambda$

$\angle LGA = 90^0 - \theta$ and AG:GB = 1:1

$\therefore$ Using the second trigonometric equation.

We have

$$(1+1)\,\text{Cot}\,(90^0 - \theta) = 1\,\text{Cot}\,(90^0 - \alpha - \lambda) - 1\,\text{Cot}\,(90^0 - \alpha + \lambda)$$

$$= \text{Cot}\,(90^0 - (\alpha + \lambda) - \text{Cot}\,(90^0 - (\alpha - \lambda)$$

(ie) $2\tan\theta = \tan(\alpha+\lambda) - \tan(\alpha-\lambda)$

$$= \frac{\text{Sin}(\alpha+\lambda)}{\text{Cos}(\alpha+\lambda)} - \frac{\text{Sin}\,(\alpha-\lambda)}{\text{Cos}(\alpha-\lambda)}$$

$$= \frac{\operatorname{Sin}(\alpha+\lambda)\operatorname{Cos}(\alpha-\lambda)-\operatorname{Cos}(\alpha+\lambda)\operatorname{Sin}(\alpha-\lambda)}{\operatorname{Cos}(\alpha+\lambda)\operatorname{Cos}(\alpha-\lambda)}$$

$$= \frac{\operatorname{Sin}[(\alpha+\lambda)-(\alpha-\lambda)]}{\operatorname{Cos}(\alpha+\lambda)\operatorname{Cos}(\alpha-\lambda)}$$

$$\therefore \tan\theta = \frac{\operatorname{Sin} 2\lambda}{2\operatorname{Cos}(\alpha+\lambda)\operatorname{Cos}(\alpha-\lambda)}$$

$$= \frac{\operatorname{Sin} 2\lambda}{\operatorname{Cos}(\alpha+\lambda+\alpha-\lambda)+\operatorname{Cos}[(\alpha+\lambda)-(\alpha-\lambda)]}$$

$$\therefore \tan\theta = \left(\frac{\operatorname{Sin} 2\lambda}{\operatorname{Cos}2\alpha+\operatorname{Cos}2\lambda}\right)$$

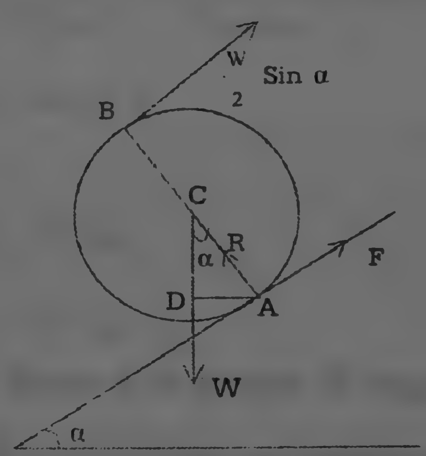
$$\theta = \tan^{-1}\frac{\operatorname{Sin} 2\lambda}{\operatorname{Cos}2\alpha+\operatorname{Cos}2\lambda}$$

Example: 5

A uniform sphere is held in equilibrium on a rough inclined plane of angle α by a force of magnitude $\frac{w}{2}$ Sinα applied tangentially to its circumference, where w in the weight of the sphere. Prove that the force must act parallel to the plane and that the coefficient of friction must be not less than $\frac{1}{2}\tan\alpha$.

Solution:

Let C be the centre of the sphere. A the point of contact of the sphere with inclined plane and AB the diameter through A.



Let the radius a.

The forces acting on the sphere are:

1) its weight W acting vertically downwards at C

2) the normal reaction R acting along AC.

3) Frictional force F acting upwards along the inclined plane as the sphere is on the point of moving down and

Space for Hints

4) The force $\frac{W \sin\alpha}{2}$ applied tangentially.

Let X be the $\perp^r$ distance of A from the line of action of $\frac{W}{2}\sin\alpha$

For equilibrium, taking moments about A

$\frac{W}{2}\sin\alpha = W.\ AD = W.\ a \sin\alpha$

(ie) $\therefore\ x = 2a$.

$\therefore$ The force $\frac{W \sin\alpha}{2}$ must be at a distance 2a from A. So it must act at B||to the plane.

Resolving along and $\perp^r$ to the plane,

We have

$$F + \frac{1}{2} W \sin\alpha = W.a \sin\alpha$$

$$\text{(ie) } F = \frac{W \sin\alpha}{2} \qquad (1)$$

$$\text{And } R = W\cos\alpha \qquad (2)$$

$\therefore\ \frac{F}{R} = \frac{W}{2}\sin\alpha \div W\cos\alpha = \frac{\tan\alpha}{2}$

For equilibrium $\frac{F}{R}$ must be $< \mu$

(ie) $\frac{\tan\alpha}{2} < \mu$

(or) $\mu > \frac{\tan\alpha}{2}$.

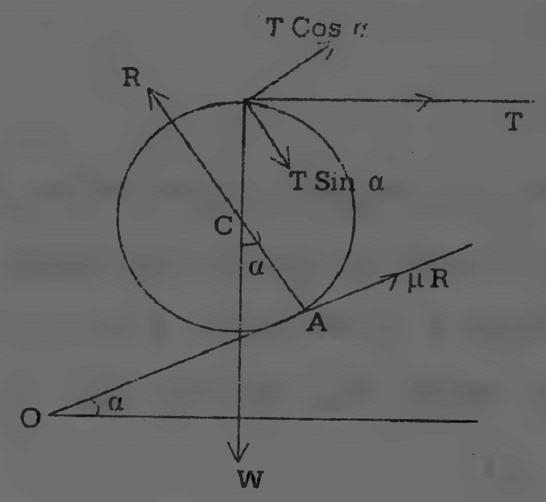
Example: 6

A sphere of weight W resting on a rough inclined plane at an angle α to the horizontal is maintained in equilibrium by a horizontal string attached to the highest point of the sphere. Show that the angle of friction in greater than $\frac{1}{2}\alpha$ and that the tension of the string is $W \tan\frac{1}{2}\alpha$

Solution:

Let the sphere rest on the inclined plane, touching it at A.

At the highest point on the sphere a string is attached and the sphere is pulled parallel to the horizontal plane.



Let $\vec{T}$ be the tension in the string. Its components along and $\perp^r$ to the inclined plane are $\perp$ $T\cos\alpha \, T\sin\alpha$.

Let us consider the case of limiting equilibrium.

Resolving the forces along and $\perp^r$ to the inclined plane.

We get

$$T\cos\alpha + \mu R = W\sin\alpha$$

And $T\sin\alpha + W\cos\alpha = R$

These give $T\cos\alpha + \mu(T\sin\alpha + W\cos\alpha) = W\sin\alpha$

$$\text{(Or) } T = W\frac{\sin\alpha - \mu\cos\alpha}{\cos\alpha + \mu\sin\alpha} = W\frac{\sin(\alpha-\lambda)}{\cos(\alpha-\lambda)} \text{ as } \mu = \tan\lambda$$

Also taking moments about A.

We get

$$T(a + a\cos\alpha) = W.a\sin\alpha$$

$$\text{(Or) } T = W\frac{\sin\alpha}{1+\cos\alpha} = W.\tan\frac{\alpha}{2}$$

Equating the two values of T

We get

$$W\frac{\sin(\alpha-\lambda)}{\cos(\alpha-\lambda)} = W.\tan\frac{\alpha}{2}$$

$$\text{(Or) } \tan(\alpha-\lambda) = \tan\frac{\alpha}{2}$$

(ie) $(\alpha-\lambda) = \frac{\alpha}{2}$ or $\lambda = \frac{1}{2}\alpha$

Space for Hints

(for limiting equilibrium)

For equilibrium,

$$\therefore \lambda > \frac{1}{2}\alpha$$

Example: 7

A uniform rod of length 2ℓ rests within a hollow sphere of radius a in a vertical plane through the centre of the sphere. The sphere in rough, the angle of friction being λ. Prove that if $\ell <$a cos λ the greatest inclination to the horizontal at which the rod can rest is given by $(2\ell - a^2)$ Sin $\theta - a^2 \text{Sin}(\theta - 2\lambda) = O$.

Solution:

The section of the hollow sphere by the vertical plane is a circle and the rod AB rests with its lower end A and upper end B on this circle.

Let the centre of the circle be C. Then

$\angle ACB = 2\alpha$ Say.

So that $\angle CAB = \angle CBA = 90^0 - \alpha$

Let the inclination of the rod to the horizontal be θ When the rod is in limiting equilibrium. Evidently this value of θ is its greatest value.

The reactions at A and B make the same angle λ(the angle of friction) with the normal to the circle at these points.

Let the reactions meet at O. Then there being only three forces in equilibrium, the line of action of the weight $\overrightarrow{W}$ of the rod acting at its midpoint G must pass through O.

(ie) OG is vertical.

Hence $\angle OGB = 90^0 - \theta$

In Δ OAB, $\angle OAB = (90 - \alpha) - \lambda$

As $\angle CAB = 90^0 - \alpha$ and $\angle OBA = 90^0 - \alpha + \lambda$

Hence applying the trigonometrical result to the Δ OAB,

We get

(AG + GB) Cot $(90^0 - \theta)$ = GB Cot $(90^0 - \alpha - \lambda)$ – AG Cot $(90^0 - \alpha - \lambda)$

(Or) $2\ \ell \tan\theta = \ell[\tan(\alpha + \lambda) - \tan(\alpha - \lambda)]$.

Space for Hints

(ie) $\tan\ \theta = \dfrac{\operatorname{Sin} 2\lambda}{\operatorname{Cos} 2\alpha + \operatorname{Cos} 2\lambda}$ on Simplication

But $\operatorname{Sin}\ \alpha = \dfrac{\ell}{\alpha}$,

So that $\operatorname{Cos} 2\ \alpha = 1 - 2\operatorname{Sin}^2 \alpha = 1 - \dfrac{2\ell^2}{a^2}$

Hence $\dfrac{\operatorname{Sin}\theta}{\operatorname{Cos}\theta} = \dfrac{\operatorname{Sin} 2\lambda}{1 - 2\ell^2/a^2 + \operatorname{Cos} 2\lambda}$

(Or) $\operatorname{Sin}\ \theta(a^2 - 2\ell^2 + a^2 \operatorname{Cos} 2\lambda) = a^2 \operatorname{Cos}\theta \operatorname{Sin} 2\lambda$

Giving $(2\ell^2 - a^2)\operatorname{Sin}\theta - a^2(\operatorname{Sin}\theta \operatorname{Cos} 2\lambda - \operatorname{Cos}\theta \operatorname{Sin} 2\lambda) = O$

(Or) $(2\ell - a^2)\operatorname{Sin}\theta - a^2 \operatorname{Sin}(\theta - 2\lambda) = O$

Note that in this position

$\lambda < 90^0 - \alpha$ Which gives $\operatorname{Cos}\lambda < \dfrac{1}{a}$ or $< a \operatorname{Cos}\lambda$

(Or) $\ell < a \operatorname{Cos}\lambda$.

Example: 8

A heavy circular disc whose plane is vertical is kept at rest on a rough inclined plane by a string parallel to the plane and touching the circle. Show that the disc will slip on the plane if the coefficient of friction be less than $\dfrac{1}{2}$ tan a where a is the inclination of the plane.

Solution:

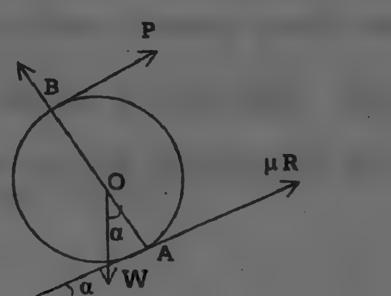
Let the circular disc centre O, be kept at rest by pulling the string in a direction parallel to the plane, as shown in the diagram.

Resolving the force that act on the disc along and $\perp^r$ to the plane.

We get

R = W Cos a and W sin a = P + μR

Taking moments about A,



Space for Hints

We get

$$P.AB = W.OA \text{ Sin } a$$

$$P = \frac{1}{2} W.\text{Sin } a$$

The disc will slip (slide) if the component W sin a of the weight of the disc acting down the plane: exceeds frictional and applied forces acting up the plane.

Hence the condition for the disc to slip in

$$W \text{ Sin} a > \mu R + P.$$

Or $W \text{ Sin } a > \mu . W \text{ Cos } a + \frac{1}{2} W.\text{Sin } a$

Giving $\tan a > 2\mu$

$$\text{(or) } \mu < \frac{1}{2} \tan a$$

Exercises:

1. How high can a particle rest inside a hollow sphere of radius a if the coefficient of friction be $\frac{1}{\sqrt{3}}$?

2. A uniform rod rests in a vertical plane within a fixed hemispherical bowl radius is equal to the length of the rod. If μ be the coefficient of friction between the rod and the bowl. Show that , in limiting equilibrium, the inclination of the rod to the horizontal is $\tan^{-1}\left(\frac{4\mu}{3-\mu^2}\right)$

3. A heavy uniform rod rests with its ends on the interior of a rough vertical circle and subtends a right angle at the centre. Show that in the position of limiting equilibrium, the rod is inclined to the horizontal at an angle equal to double the angle of friction.

Example: 9

A square lamina whose plane is vertical rests with the ends of a side against a rough vertical wall and a rough horizontal ground. If the coefficients of friction for the ground and the wall be μ and μ' respectively

prove that, when lamina is on the point of motion, the inclination of the side in question to the horizontal is $\tan^{-1}\dfrac{1-\mu\mu'}{1+2\mu+\mu\mu'}$

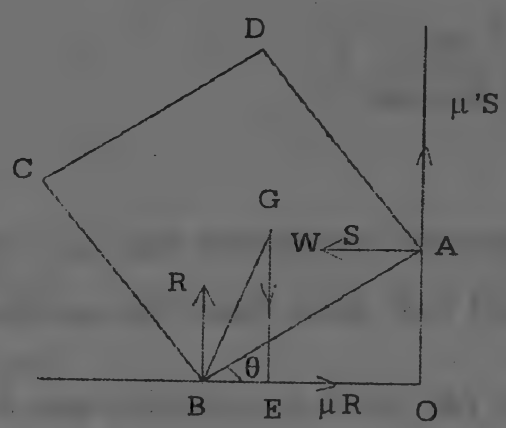
Solution:

Let ABCD be the square lamina and the side AB (=a) be in a contact with the at A and the ground at B. The forces acting on the lamina are:

i) normal reaction R at B $\perp^r$ to the ground

ii) Limiting friction μR at B as shown in the figure.

iii) Normal reaction S at A $\perp^r$ to the wall.



iv) Limiting friction μ' S along the wall upwards and

v) Its weight W acting vertically downwards at G it centre of gravity.

Let AB make an angle θ to the horizontal

Resolving horizontally and vertically,

$$S = \mu R \qquad (1)$$

$$R + \mu' S = W \qquad (2)$$

Taking the moments about B

$$S.\,OA + \mu' S.OB = W.BE$$

(ie) $S.\ a \sin\theta + \mu' S.a\cos\theta = W.BG\cos(45^0+\theta)$

$$= \frac{Wa}{\sqrt{2}}(\cos 45^0 \cos\theta - \sin 45^0 \sin\theta)$$

$$= \frac{Wa}{2}(\cos\theta - \sin\theta)$$

(ie) $S\ (\sin\theta + \mu'\cos\theta - \sin\theta) = \dfrac{W}{2}(\cos\theta - \sin\theta)$ (3)

From (2) $R = w - \mu' S$ and putting this in (1)

We have $S = \mu(W - \mu' s)\mu$

$$(ie)\ S = \frac{\mu W}{1+\mu\mu'} \qquad (4)$$

Substituting (4) in (3)

$$\frac{\mu W}{1+\mu\mu'}(\text{Sin}\theta + \mu'\text{Cos}\theta) = \frac{W}{2}(\text{Cos}\theta - \text{Sin}\theta)$$

$$(ie)\ 2\ \mu(\text{Sin}\,\theta+\mu'\text{Cos}\theta)=(1+\mu\mu')(\text{Cos}\theta-\text{Sin}\,\theta)$$

$$\text{Sin}\ \theta(2\mu+1+\mu\mu')=\text{Cos}\,\theta.(1+\mu\mu'-2\mu\mu')$$

$$\therefore \tan\theta=\frac{\text{Sin}\,\theta}{\text{Cos}\,\theta}=\frac{1-\mu\mu'}{1+2\mu+\mu\mu'}$$

$$\therefore \theta=\tan^{-1}\left(\frac{1-\mu\mu'}{1+2\mu+\mu\mu'}\right)$$

Example: 10

A solid homogeneous hemisphere rests on a rough horizontal plane and against a smooth vertical wall, show that if the coefficient of friction be greater than $\frac{3}{8}$, the hemisphere can rest in any position and, if it be less, the least angle that the base of the hemisphere can make with the vertical is $\text{Cos}^{-1}\left(\frac{8\mu}{3}\right)$

If the wall be rough (Coefficient of friction μ') Show that the angle is $\text{Cos}^{-1}\left[\frac{8\mu(1+\mu\mu')}{3(1+\mu\mu')}\right]$

Solution:

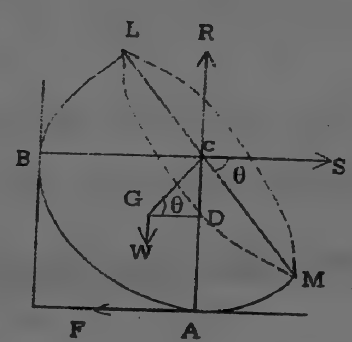


Fig. (a)

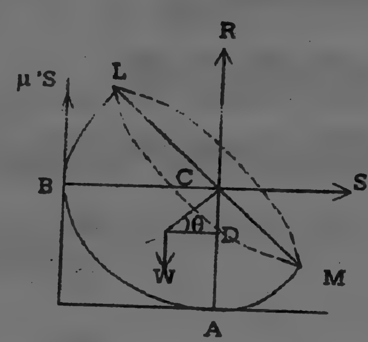


Fig. (b)

Figure represents a section of the hemisphere by the plane $\perp$r to the wall. A is the point of contact with rough horizontal plane and B with the smooth wall LM is the base and C is the centre of hemisphere.

The centre of gravity is at G on the radius $\perp$r to LM. Such the

$CG = \frac{3r}{8}$, r being the radius of the hemisphere

The forces acting are,

i) reaction R $\perp^r$ to the ground at A

ii) friction F at A.

iii) reaction S $\perp^r$ to the smooth wall at B and

iv) Weight W of the hemisphere acting vertically downwards at G.

Let $\angle ACM = \theta$

Resolving horizontally and vertically,

$$S = F \quad (1)$$

$$R = W \quad (2)$$

Taking moments about A,

$$S.\,CA = W.GD$$

$$= W.CG \cos\theta$$

$$S.r = W.\frac{3w}{8}\cos\theta$$

$$\text{(Or) } S = \frac{3w}{8}\theta \quad (3)$$

From (2) and (3)

$$F = \frac{3w\cos\theta}{8} \text{ and } R = w \text{ from (2)}$$

$$\therefore \frac{F}{R} = \frac{3w\cos\theta}{8W} = \frac{3\cos\theta}{8}$$

For equilibrium,

$\frac{F}{R}$ must be $\leq \mu$, the coefficient of friction.

(ie) $\frac{3\cos\theta}{8} \leq \mu$,

$$\text{(Or) } \mu \geq \frac{3}{8}\cos\theta \quad (4)$$

If μ is $\frac{3}{8}$, then as $\cos\theta$ is < 1, μ will automatically $\geq \frac{3}{8}\cos\theta$.

Hence condition (4) is satisfied and the hemisphere can rest in any position.

Space for Hints

From (4) $\cos\theta \le \frac{8\mu}{3}$.

If $\mu < \frac{3}{8}, \frac{8\mu}{3} < 1$

Let α be the angle such that $\cos\alpha = \frac{8\mu}{3}$

Then $\cos\theta \le \cos\alpha$ for equilibrium,

(ie) $\theta \ge \alpha$

(or) $\theta \ge \cos^{-1}\frac{8\mu}{3}$

Hence the least value of θ is $\cos^{-1}\left(\frac{8\mu}{3}\right)$

When the wall is rough, limiting frictions μR and $\mu' S$ act at A and B refer to figure.

Resolving horizontally and vertically.

$$S = \mu R \qquad (5)$$

$$R + \mu' S = W \qquad (6)$$

Taking moments about A,

$$S.CA + \mu' S.BC = W.GD$$

(Or) $$Sr + \mu' S.\, r = W.\frac{3r}{8}\cos\theta$$

$$S(1+\mu') = \frac{3W}{8}\cos\theta \qquad (7)$$

Putting the value of from (6) in (5)

We have

$$S = \mu(W - \mu' S)$$

(ie) $$S = \frac{\mu W}{1+\mu\mu'} \qquad (8)$$

Substituting (8) in (7) θ

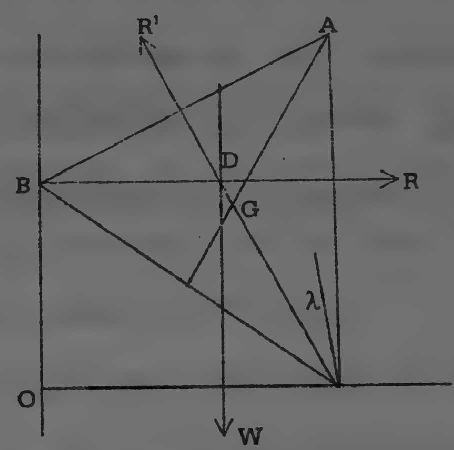
We have $$\frac{\mu W}{1-\mu\mu'} \quad (1+\mu') = \frac{3W}{8}\cos\theta$$

(ie) $\cos\theta = \frac{8\mu(1+\mu')}{3(1+\mu\mu')}$ Which gives the least angle θ

Example: 11

A thin equilateral triangular plate of uniform density rests in a vertical plane with one end of its base on a rough horizontal floor and the other end against a smooth vertical wall. Show that the least angle its base can make with the horizontal is given by $\cot\theta = 2\mu + \frac{1}{\sqrt{3}}$. Where μ is the coefficient of friction.

Solution:



Let ABC be the plate of weight $\vec{W}$ with its one vertex B on the smooth vertical wall and another end C on the horizontal floor.

The reaction $\vec{R}$ at B. is $\perp^r$ to OB. Let the weight $\vec{W}$ of the plate acting through G meet $\vec{R}$ at D. Then the reaction $\vec{R}'$ at C must pass through D, there being equilibrium.

Since the equilibrium is limiting. R' makes the angle λ (the angle of friction with the vertical at C.

Let $\angle BCO = \theta$.

From Δ BCD, $\frac{BD}{\sin\angle BCD} = \frac{BC}{\sin\angle BDC}$

(Or) $$\frac{BD}{\sin(90^0 - \theta - \lambda)} = \frac{BC}{\sin(90^0 + \lambda)} \text{ as } \angle CDG = \lambda$$

(ie) $$BD = BC\,\frac{\cos(\theta+\lambda)}{\cos\lambda}$$

But $BD = BG\cos\angle GBD = \frac{1}{\sqrt{3}} BC\cos(\theta - 30^0)$

As $\angle GBD = \angle CBD - \angle CBG$

Hence $$\frac{1}{\sqrt{3}} BC\ \cos(\theta - 30^0) = BC\frac{\cos(\theta+\lambda)}{\cos\lambda}$$

Space for Hints

(or) $\frac{1}{\sqrt{3}}\left[\operatorname{Cos}\theta\frac{\sqrt{3}}{2}+\operatorname{Sin}\theta\frac{1}{2}\right]=\frac{\operatorname{Cos}\theta\operatorname{Cos}\lambda-\operatorname{Sin}\theta\operatorname{Sin}\lambda}{\operatorname{Cos}\lambda}$

Thus $\frac{1}{2}\operatorname{Cos}\theta=\operatorname{Sin}\theta\left[\tan\lambda+\frac{1}{2\sqrt{3}}\right]$

$$\text{(Or) } \operatorname{Cot}\theta=2\mu+\frac{1}{\sqrt{3}}.$$

Example: 12

A solid hemisphere rests in equilibrium on a rough ground and against an equally rough vertical, the coefficient of friction being μ. Show that if the equilibrium is limiting, the inclination of the base to the horizon is

$\sin^{-1}\left[\frac{8\mu(1+\mu)}{3(1+\mu^2)}\right]$

Solution:

Let the hemisphere of radius r have contact with the ground and the wall at A and B.

The weight reactions $\vec{R}$ and $\vec{S}$ at A and B act through the centre O of the base $\overrightarrow{\mu R}$ and $\mu\vec{S}$ act as in the diagram.

$$\angle LOB=\theta;\text{then}\angle GOC=\theta$$

Resolving the forces horizontally and vertically,

We get

$$S=\mu R \text{ and } R+\mu S=W$$

$$S=\mu(W-\mu S) \text{ giving } S=\frac{\mu w}{1+\mu^2}$$

Also taking moments about A.

We get

$$S.r+\mu S.r=W.GC=W(OG\operatorname{Sin}\theta)$$

$$=W.\frac{3}{8}r\operatorname{Sin}\theta$$

(Or) $\operatorname{Sin}\theta=\frac{8}{3W}(1+\mu)S$

$$=\frac{8}{3w}(1+\mu).\frac{\mu W}{1+\mu^2}=\left[\frac{8\mu(1+\mu)}{3(1+\mu^2)}\right]$$

Thus $\theta = \text{Sin}^{-1}\left[\dfrac{8\mu(1+\mu)}{3(1+\mu^2)}\right]$

Exercises:

1. A square Lamina rests in a vertical plane with one end of one side against a rough wall and the other end against an equally rough horizontal ground. Show that the edge of the lamina in question in inclined to the horizontal at angle $\tan^{-1}\left(\dfrac{1-\mu}{1-\mu}\right)$ where μ is the coefficient of friction at each end.

2. A Uniform thin hemispherical bowl rests with its curved surface on a rough horizontal plane (coefficient of friction μ) and leans against a smooth vertical wall, prove that when the bowl is on the point of slipping the inclination of the axis of the bowl to the vertical in $\sin^{-1}(2\mu)$.

3. A hemispherical shell rests with its curved surface on a rough plane. Whose angle of frictions is λ. Show that the inclination of the plane base of the rim to the horizon cannot be greater then $\text{Sin}^{-1}(2\ \text{Sin}\ \lambda)$.

Problems on cone of Friction:

Example: 1

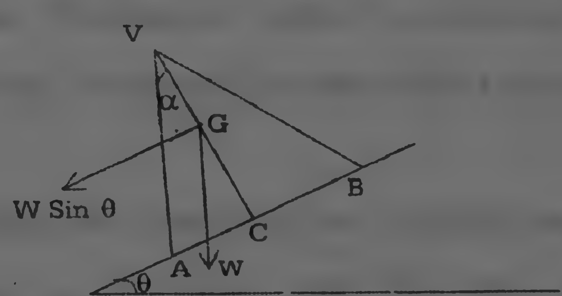
A uniform cone is placed on a rough inclined plane. As the inclination of the plane increases. Show that the cone will slide before it topples over if the coefficient of friction is less then 4 tan α. where α. is the semi – vertical angle of the cone.

If $\dfrac{1}{\sqrt{3}}$ be the coefficient of friction, find the angle of the cone when its is on the point of both slipping and turning over.

Solution:

Let AVB represent the section of the cone by a vertical plane through its centre of gravity g.

Then $GC = \dfrac{1}{4}VC$.



Space for Hints

Space for Hints

When the inclination of the plane in θ. Let the cone be on the point of sliding.

The resultant normal reaction $\vec{R}$ acts at a point on AB (the whole of AB being is contact with plane) and so for sliding. We must have W Sin $\theta > \mu R$

But resolving $\perp^r$ to the plane.

We get $R = W \cos\theta$.

Hence the cone will slide if $W \sin\theta > \mu . W \cos\theta$

(Or) $\tan\theta > \mu$ (1)

In the case when toppling in about to take place (which can happen only about A), the normal reaction and the frictional force act through A.

Hence on taking moments about A,

We get the condition for the cone to topple $W \sin\theta . GC > W \cos\theta . AC$

(Or) $\tan\theta > 4\tan\alpha$ (or) $\tan\theta > \frac{AG}{GC} = 4\frac{AG}{VC}$ as $GC = \frac{1}{4}VC$ (2)

From (1) and (2)

We find the cone will slide before it topples over if $4\tan\alpha < \mu$ or $\mu < 4\tan\alpha$.

Sliding and toppling over will occur simultaneously if $\mu = 4\tan\alpha$

But $\mu = \frac{1}{\sqrt{3}}$

Hence $4\tan\alpha = \frac{1}{\sqrt{3}}$ (or) $\alpha = \tan^{-1}\frac{1}{4\sqrt{3}}$

$\therefore$ The angle of cone is

$$2\alpha = 2\tan^{-1}\frac{1}{4\sqrt{3}}$$

Example: 2

A heavy solid right circular cone is placed with its base on a rough inclined plane, the inclination of which is gradually increased. Determine whether the initial motion of the cone will be one of sliding or tumbling.

Solution:

As the previous problem, we find that if the angle of inclination θ is steadily increased, the cone will slide initially if $\mu < \tan\theta$ (ie) sliding will occur before toppling over.

Space for Hints

So long as $\mu < 4\tan\alpha$, where α in the semi vertical angle of the cone there will be sliding only.

If θ is such that tan θ = 4 tanα, sliding and toppling over occur simultaneously.

Exercise:

1. A cone of vertical angle 2α, rests with its base on a rough plane inclined to the horizon. As the inclination of the plane is gradually increased, show that the cone will slide before it topples over, if the coefficient of friction be less than 4 tan α.

Show that it will slide topples over if the coefficient of friction is $< \frac{4r}{h}$, r being the base radius & h the height what happens if the coefficient.

2. A uniform solid cone with circular base in placed with its plane base on a rough inclined plane and the inclination to the horizon in gradually in increased. Find the ratio of the radius of the base of the cone to its height if the cone is to topple before it slides. Coefficient of friction = μ.

3. A uniform cone, resting with its circular base in contact with a rough inclined plane (angle α) is on the point of both slipping and toppling. If the angle of friction be λ, find α and the vertical angle of the cone in terms of λ.

4. A right cone is placed with its base on a rough inclined plane. If $\frac{1}{\sqrt{3}}$ be the coefficient of friction, find the angle of the cone when it is on the point of both slipping and turning over.

Space for Hints

DYNAMICS

UNIT – 6

PROJECTILES

6.1 Introduction:

Consider motion of a particle projected into the air in any direction and with any velocity. Such a particle is called a projectile.

The two forces that act on the projectile are its weight and the resistance of air.

For simplicity, we suppose the motion to take place with such a moderate distance from the surface of the surface of the earth that we can neglect the variations in the acceleration due to gravity. This means that g may be considered to be constant in magnitude throughout the motion of the projectile. Secondly we shall neglect the resistance of the air and consider the motion to take place in vacuum.

6.1.1. Definitions:

The following terms are used in connection with projectiles.

The angle of projection is the angle that the direction in which the particle is initially projected makes with the horizontal plane through the point of projection.

The velocity of projection is the velocity with which the particle is projected.

The trajectory is the path which the particle describes.

The Range on a plane through the point projection is the distance between the point of projection and the point where the trajectory meets that plane.

The time of flight is the interval of time that elapses from the instant of projection till the instant when the particle again meets the horizontal plane through the point of projection.

6.2 PATH OF PROJECTILE

Theorem: Show that the path of a projectile is a parabola.

Proof:

Let a particle be projected from O with a velocity u at an angle α to the horizon.

Space for Hints

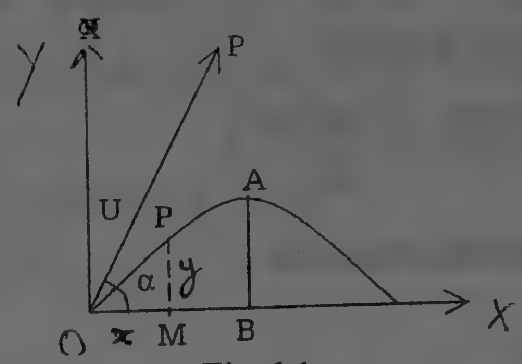


Fig.6.1

Take O as the origin, the horizontal and the upward vertical through O as axes of X and Y respectively. The initial velocity u can be split into two components, which are u$\cos\alpha$ in the horizontal direction and u$\sin\alpha$ in the vertical direction. The horizontal component. u$\cos\alpha$ is constant throughout the motion as there is no horizontal acceleration. The vertical component u sin α is subject to an acceleration of downwards.

Let P (x, y) be the position of the particle at time t secs after projection.

Then

x = horizontal distance described in t secs = (u cos α).t (1)

y = vertical distance described in t secs = (u cos α) t – 1/2g^2 (2)

(1) and (2) can be taken as the parametric equations of the trajectory. The equation to the path is got by eliminating t between them

From (1) $$t = \frac{x}{u \operatorname{Cos}\alpha}$$

And putting this in (2)

We get $$y = u\operatorname{Sin}\alpha \frac{x}{u\operatorname{Cos}\alpha} - \frac{1}{2} g\left(\frac{x}{u\cos\alpha}\right)^2$$

$$Y = x \tan\alpha - \frac{gx^2}{2u^2 \operatorname{Cos}^2\alpha} \qquad (3)$$

Multiplying (3) by $2u^2 \operatorname{Cos}^2 \alpha$.

$$2u^2 \operatorname{Cos}^2 \alpha.y = 2u^2 \operatorname{Cos}^2 \alpha.x \frac{\operatorname{Sin}\alpha}{\operatorname{Cos}\alpha} - gx^2$$

$$\text{(ie)}x^2 - \frac{2u^2 \operatorname{Sin}\alpha \operatorname{Cos}\alpha}{g} x = -\frac{2u^2 \operatorname{Cos}^2\alpha}{g} y.$$

$$\left(x - \frac{u^2 \operatorname{Sin}\alpha \operatorname{Cos}\alpha}{g}\right)^2 = \frac{u^4 \operatorname{Sin}\alpha \operatorname{Cos}^2\alpha}{g^2} - \frac{2u^2 \operatorname{Cos}^2\alpha}{g} y.$$

$$= -\frac{2u^2 \operatorname{Cos}^2\alpha}{g}\left(y - \frac{u^2 \operatorname{Sin}^2\alpha}{2g}\right)$$

Transfer the origin to the point

$$\left(\frac{u^2 \operatorname{Sin}\alpha \operatorname{Cos}\alpha}{g}, \frac{u^2 \operatorname{Sin}^2 \alpha}{2g}\right)$$

The above equation then becomes

$$X^2 = -\frac{2u^2 \operatorname{Cos}^2\alpha}{g}.Y \qquad (4)$$

Where $X = x - \dfrac{u^2 \operatorname{Sin}\alpha \operatorname{Cos}\alpha}{g}$

$$Y = y - \frac{u^2 \operatorname{Sin}\alpha^2}{2g}.$$

(4) is clearly the equation to a parabola latus rectum $\dfrac{2u^2 \operatorname{Cos}^2\alpha}{g}$

whose axis is vertical and downwards and whose vertex is the point

$$\left(\frac{u^2 \operatorname{Sin}\alpha \operatorname{Cos}\alpha}{g}, \frac{u^2 \operatorname{Sin}^2 \alpha}{2g}\right)$$

Note:

The latus rectum of the above parabola is

$$= \frac{2u^2 \operatorname{Cos}^2\alpha}{g} = \frac{2}{g}(u \operatorname{Cos}\alpha)^2$$

$$= \frac{2}{g} x \text{ (Square of the horizontal velocity).}$$

So the latus rectum (ie) the size of the parabola) is independent of the initial vertical velocity and depends only on the horizontal velocity.

6.3 Characteristics of the motion of a projectile:

Refer to fig (6.1)

Let a particle be projected from O with velocity u at an angle α to the horizontal OX. Let A be the highest point of the path & C the point through O. Using the two fundamental principles given in (1) we can drive the following results relation to the motion of a projectile.

(1). Greatest height attained by a projectile.

At the highest point, the particle will be moving only horizontally, having lost all its vertical velocity.

Space for Hints

Let AB = h = the greatest height reached. Considering vertical motion separately, initial upward vertical velocity = u sin α and the acceleration in this direction is – g.

The final vertical velocity at A is = O.

Hence $O = (u\sin\alpha)^2 - 2g.h.$

(ie) $h = \dfrac{u^2\sin^2\alpha}{2g}$

(ie) The vertex of the parabola is the highest point of the path.

(2). Time taken to reach the greatest height

Let T be the time from O to A. Then, in time T, the initial velocity u sin α is reduced to zero, acted on by an acceleration – g.

Hence $O = u\sin\alpha$ - gT.

$T = \dfrac{u\sin\alpha}{g}$

(3). Time of fight (ie) the time taken to return to the same horizontal level as O. When the particle arrives at O, the effective vertical distance it has described is zero.

Hence if t is the time of flight, considering vertical motion,

We have,

$O = u\sin\alpha.t \; \dfrac{1}{2}gt^2$

(ie) $t = O$ or $t = \dfrac{2u\sin\alpha}{g}$

t = O is the instant of projection when also the vertical distance travelled is zero.

$\therefore$ The time of flight $= \dfrac{2u\sin\alpha}{g}$

We find that the time of flight is twice the time taken to reach the highest point, as we should except from symmetry.

(4). The range on the horizontal plane through the point of projection

The time of flight is $t = \dfrac{2u\sin\alpha}{g}$

During this time, the horizontal velocity remains constant and is equal to $u\cos\alpha$.

Hence

Space for Hints

OC = horizontal distance described in time t.

$$= u\cos\alpha . t = u\cos\alpha . \frac{\sin\alpha}{g}$$

$$= \frac{2u^2\sin\alpha\cos\alpha}{g}$$

Hence

The horizontal range $R = \dfrac{2u^2 \text{Sin}\alpha\text{Cos}\alpha}{g} = \dfrac{u^2\text{Sin}2\alpha}{g}$

Note:

(1) The horizontal range can also be found thus:

The equation to the path is

$$Y -= x\tan\alpha - \frac{gx^2}{2u^2\text{Cos}^2\alpha} \qquad (1)$$

The equation to the x – axis is y = O. Putting y = O in (1)

We have $x\tan\alpha - \dfrac{gx^2}{2u^2\text{Cos}^2\alpha} = O$

(ie) $x = O$ or $x = \dfrac{2u^2\text{Cos}^2\alpha\tan\alpha}{g} = \dfrac{2u^2\text{Sin}\alpha\text{Cos}\alpha}{g}$

x = O, Corresponds to the point of projection and so the other value.

$\dfrac{2u^2\text{Sin}\alpha\text{Cos}\alpha}{g}$ gives the horizontal range.

(2) Horizontal range $= \dfrac{2u^2\text{Sin}\alpha\text{Cos}\alpha}{g}$

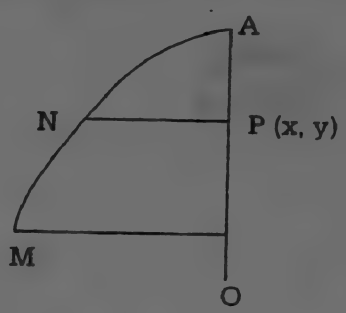
$$= \frac{2(u\text{Cos}\alpha)(u\text{Sin}\alpha)}{g}$$

$= \dfrac{2UV}{g}$ Where U and V are the initial horizontal and vertical velocities.

6.2.3 A particle is projected horizontally from a point at a certain height above the ground to show that the path described by it is a parabola.

Space for Hints

Let a particle be projected horizontally with a velocity u from a point A at a height h above the ground level. Let it strike the ground at M. Take A as origin the horizontal through A as x-axis and the downwards vertical through A as y axis. Let P(x,y) be the position of the particle at time t. As there is no horizontal acceleration, the horizontal velocity remains constant throughout the motion.



So x = horizontal distance described in time t = ut (1)

Due to gravity, the vertical acceleration during the motion is g downwards.

Y = vertical distance, described in time t = $\frac{1}{2}gt^2$ (2)

Eliminate t between (1) & (2)

We have

$$Y = \frac{1}{2}.g.\frac{x^2}{u^2} \text{ (ie) } x^2 = \frac{2u^2}{g}y$$

(3) Shows that Y is a quadratic function of X.

So it represents a parabola with vertex at A and Axis AN.

Worked examples

Example 1:

A body is projected with a velocity of 98 metres per sec in a direction making an angle $\tan^{-1}3$ with the horizon. Show that it rises to a vertical height of 441 metres and that its time of flight is about 19 secs. Find also the horizontal range through the point of projection. [$g = 9.8\ m/sec^2$]

Solution:

Here $u = 98,\ \alpha = \tan^{-1}3$ (ie) $\tan\alpha = 3$.

$$\therefore \text{Sin}\alpha = \frac{\text{Sin}\alpha}{\text{Cos}\alpha}.\text{Cos}\alpha = \frac{\tan\alpha}{\sec\alpha} = \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}$$

$$= \frac{3}{\sqrt{1+9}} = \frac{3}{\sqrt{10}}$$

$$\text{Cos}\alpha = \frac{\text{Sin}\alpha}{\tan\alpha} = \frac{1}{\sqrt{10}}$$

$$\text{Greatest height reached} = \frac{u^2 \sin^2\alpha}{g} = \frac{98x98x9}{10x2x9.8}$$

= 441 metres

$$\text{Time of flight} = \frac{2u\sin\alpha}{g} = \frac{2x98x3}{\sqrt{10}x9.8}$$

$= 6\sqrt{10} = 6 \text{ x } 3.162$

= 18.972

= 19 secs, nearly

$$\text{Horizontal range} = \frac{2u^2\text{Sin}\alpha\text{Cos}\alpha}{g}$$

$$= \frac{2x98x3}{9.8} \text{x} \frac{3}{\sqrt{10}} \text{x} \frac{1}{\sqrt{10}}$$

= 588 metres.

Example: 2

If the greatest height attained by the particle in a quarter of its range on the horizontal plane through the point of projection, find the angle of projection.

Solution:

Let u be the initial velocity and α the angle of projection.

$$\text{Then, the greatest height} = \frac{u^2\text{Sin}^2\alpha}{2g}$$

$$\text{And horizontal range} = \frac{2u^2\text{Sin}\alpha\text{Cos}\alpha}{g}$$

$$\text{It is given that } \frac{u^2\text{Sin}^2\alpha}{2g} = \frac{1}{4} \text{x} \frac{2u^2\text{Sin}\alpha\text{Cos}\alpha}{g}$$

$$\text{(ie) } \frac{u^2\text{Sin}^2\alpha}{2g} = \frac{u^2\text{Sin}\alpha\text{Cos}\alpha}{2g}$$

(ie) $\text{Sin}\alpha = \text{Cos}\alpha \text{(or)} \tan\alpha = 1$

$\therefore \alpha = 45^\circ$.

Example: 3

A stone in thrown with a velocity of 39.2 m/sec at 30° to the horizontal. Find at what times it will be at height of 14.7 ($g = 9.8 \text{ m/sec}^2$).

Space for Hint

Initial vertical velocity = 39.2 x Sin30°

= 19.6 m/sec.

This is subject to an acceleration – g.

Let the particle be at a height 14.7m after time t sec.

Applying the formula "$s = ut + \frac{1}{2}at^2$"

We have $14.7 = 19.6 - \frac{1}{2}gt^2$

$= 19.6t - 4.9t^2$

(ie) $3 = 4t - t^2$ or $t^2 - 4t + 3 = 0$

(ie) $(t-3)(t-1) = 0$; $t = 1$ or $t = 3$.

Hence at the end of 1 sec. And again at the end of 3 secs.

It will be at a height of 14.7m

Example: 4

A bomb was released from an aeroplane when it was at a height of 1960m. above a point A on the ground and was moving horizontally with a speed of 100m per sec. Find the distance from A of the point where the bomb strikes the ground. (g=9.8m/sec^2).

Solution:

Let us consider the motion of the bomb in the horizontal and the vertical directions separately. The dynamical details of each motion may be presented as follows:

Horizontal Motion	Vertical Motion (upwards +ve)
Initial velocity = 10m/Sec	Initial velocity = 0
Acceleration = O	Acceleration = -9.8m/Sec^2
	Distance = -1960m ($\because$ It is downwards)

Let t be the time taken by the bomb to strike the point on the ground. Consider the vertical motion, and applying the formula.

$$"S = ut + \frac{1}{2}at^2"$$

We have $-1960 = O\,xt - \frac{1}{2}x9.8t^2$

(ie) $4.9t^2 = 1960 \Rightarrow t^2 = 400$. (ie) $t = 20$ secs.

During this time, the horizontal velocity is constant.

$\therefore$ The horizontal distance described by the bomb in 20 secs = 20 x 100

Space for Hints

$= 2000$m.

Example: 5

A particle is projected so as to graze the tops of two parallels walls, the first of height 'a' at a distance to from the point of projection and the second of height b at a distance 'a' from the point of projection. If the path of the particle lies in a plane $\perp^r$ to both the walls, find the range on the horizontal plane and show that the angle of projection exceeds $\tan^{-1}3$.

Solution:

u being initial velocity and α the angle of projection, the equation to the path is

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$

$$\text{(ie) } y = xt - \frac{gx^2}{2u^2}\left(1+t^2\right) \text{ where } t = \tan\alpha \qquad (1)$$

The tops of the two walls are (b, a) and (a, b) respectively and they lie on (1)

$$\therefore a = bt - \frac{gb^2}{2u^2}\left(1+t^2\right) \qquad (2)$$

$$\& \; b = at - \frac{ga^2}{2u^2}\left(1+t^2\right) \qquad (3)$$

$$\text{From (2), } a - bt = -\frac{gb^2}{2u^2}\left(1+t^2\right) \qquad (4)$$

$$\text{From (3), } b - at = -\frac{ga^2}{2u^2}\left(1+t^2\right) \qquad (5)$$

Divide (4) by (5)

We get $\dfrac{a-bt}{b-at} = \dfrac{b^2}{a^2}$

(ie) $b^3 - ab^2t = a^3 - a^2bt$

$t\left(a^2b - ab^2\right) = a^3 - b^3$

$$\therefore t = \frac{a^3 - b^3}{a^2b - ab^2} = \frac{(a-b)\left(a^2+ab+b^2\right)}{ab(a-b)}$$

$$t = \frac{a^2+ab+b^2}{ab}$$

$$\tan\alpha = \frac{a^2+ab+b^2}{ab} = \frac{\left(a^2-2ab+b^2\right)+3ab}{ab}$$

$$\therefore \tan\alpha = \frac{(a-b)^2}{ab} + 3 \qquad (6)$$

The first term in the right side of (6) is positive

$\therefore \tan\alpha > 3$ or $\alpha > \tan^{-1} 3$

From (4) $\dfrac{g(1+t^2)}{2u^2} = \dfrac{a-bt}{-b^2} = \dfrac{bt-a}{b^2}$

$$= \frac{\dfrac{b(a^2+ab+b^2)-a}{ab}}{b^2} = \frac{a^2+ab+b^2-a^2}{ab^2} = \frac{b(a+b)}{ab^2} = \frac{a+b}{ab} \qquad (7)$$

Horizontal range $R = \dfrac{u^2 \mathrm{Sin}2\alpha}{g} = \dfrac{2u^2 t}{g(1+t^2)}$

$= t.\dfrac{ab}{a+b}$ Substituting from (7)

$= \dfrac{a^2+ab+b^2}{ab}.\dfrac{ab}{a+b}$

$= \dfrac{a^2+ab+b^2}{a+b}$

Example: 6

A particle is thrown over a triangle from one end of a horizontal base and grazing the vertex falls on the other end of the base. If A, B are the base angles, and α the angle of projection, show that $\tan\alpha = \tan A + \tan B$.

Solution:

Let u be the velocity and α the angle of projection and let t secs be the time from A to C.

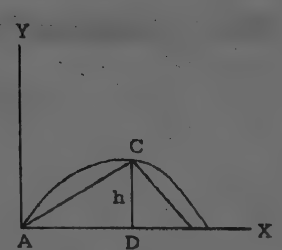
Draw CD $\perp$ AB and Let CD = h.

Considering vertical motion,

h = vertical distance described in time t.

$= u \sin\alpha .t - \dfrac{1}{2}gt^2$

AD = horizontal distance described in time t = $u\mathrm{Cos}\,\alpha .t$



Space for Hints

From ΔCAD,

$$\tan A = \frac{CD}{AD} = \frac{h}{AD} = \frac{uSin\alpha.t - \frac{1}{2}gt^2}{u\cos\alpha.t}$$

$$\tan A = \tan\alpha - \frac{gt}{2uCos\alpha} \qquad (1)$$

AB = Horizontal range

$$= \frac{2u^2 Sin\alpha\cos\alpha}{g}$$

$$\therefore\ DB = AB - AD$$

$$= \frac{2u^2 Sin\alpha Cos\alpha}{g} - uCos\alpha.t$$

From Δ CDB

$$\tan B = \frac{CD}{DB} = \frac{h}{\left(\frac{2u^2 Sin\alpha Cos\alpha}{g} - uCos\alpha.t\right)}$$

$$= \frac{uSin\alpha.t - \frac{1}{2}gt^2}{\left(\frac{2u^2 Sin\alpha Cos\alpha}{g} - uSin\alpha.t\right)}$$

$$= \frac{gt\left(2uSin\alpha - \frac{1}{2}gt\right)}{uCos\alpha(2uSin\alpha - gt)}$$

$$\tan B = \frac{gt(2uSin\alpha - gt)}{2uCos\alpha(2uSin\alpha - gt)} = \frac{gt}{2uCos\alpha} \qquad (2)$$

Adding (1) and (2)

$$\tan A + \tan B = \tan\alpha - \frac{gt}{2uCos\alpha} + \frac{gt}{2uCos\alpha}$$

$$\therefore\ \tan A + \tan B = \tan\ \alpha.$$

Example: 7

Show that the greatest height which a particle with initial velocity v can reach on a vertical wall a distance 'a' from the point of projection is

$$\frac{v^2}{2g} - \frac{ga^2}{2v^2}$$

Prove also that the greatest height above the point of projection attained by the particle in its flight is $v^6/2g\,(v^4 + g^2a^2)$.

Solution:

In the usual notation, the equation to the path is

$$y = x \tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha} \qquad (1)$$

Putting x = a in (1)

We get the value of y, which is the height reached on the vertical wall at a distance 'a' from the point of projection.

$$\therefore y = a\tan\alpha - \frac{ga^2}{2v^2\cos^2\alpha}$$

$$= at - \frac{ga^2}{2v^2}\left(1+t^2\right) \text{ where } t = \tan\alpha \qquad (2)$$

Now a and v are given and so y in function of t.

$\therefore$ y is maximum when $\frac{dy}{dt} = 0$ and $\frac{d^2y}{dt^2}$ is negative.

Differentiating (2) with respect to t.

$$\frac{dy}{dt} = a - \frac{ga^2}{2v^2}.2t = a - \frac{ga^2t}{v^2}$$

$$\frac{d^2y}{dt^2} = -\frac{ga^2}{V^2} = \text{negative.}$$

So y is maximum when $a - \frac{ga^2t}{v^2} = O$ or $t = \frac{v^2}{ga}$ (3)

Putting this value of t in (2).

$$\text{Max. Value of } y = a.\frac{v^2}{ga} - \frac{ga^2}{2v^2}\left(1 + \frac{v^4}{g^2a^2}\right)$$

$$= \frac{v^2}{2g} - \frac{ga^2}{2v^2} - \frac{v^2}{2g}$$

$$= \frac{v^2}{2g} - \frac{ga^2}{2v^2}$$

This is the greatest height reached on the wall.

Greatest height attained during the flight.

Space for Hints

Space for Hints

$$= \frac{v^2 \text{Sin}^2\alpha}{2g} = \frac{v^-}{2g} \cdot \frac{1}{\text{Co sec}^2 \alpha} = \frac{v^2}{2g(1+\text{Cot}^2\alpha)}$$

$$= \frac{v^2}{2g\left(1+\frac{g^2a^2}{v^4}\right)}$$ putting the value of tan α from (3)

$$= \frac{v^6}{2g(v^4+g^2a^2)}$$

Example: 8

Show that the greatest height attained by a particle projected with a velocity u at an angle α to the horizontal in unaltered. If the velocity of projection is increased to ku and the angle of projection α is decreased by λ where,

$\text{Co sec}\,\lambda = k(\text{Cot}\lambda - \text{Cot}\alpha)$

Solution:

The greatest height attained by the particle when the velocity projection is u and the angle of projection is α given by $\frac{u^2\text{Sin}^2\alpha}{2g}$.

It will be given by $\frac{(ku)^2\text{Sin}^2(\alpha-\lambda)}{2g}$ when the velocity of projection is increased to ku and the angle or projection decreased by λ. These two are equal it.

$$\frac{u^2\text{Sin}^2\alpha}{2g} = \frac{(ku)^2\,\text{Sin}^2(\alpha-\lambda)}{2g}$$

(ie) if $\text{Sin}^2\,\alpha = k^2\,\text{Sin}^2\,(\alpha-\lambda)$.

(ie) if $\frac{\text{Sin}\,(\alpha-\lambda)}{\text{Sin}\,\alpha} = \frac{1}{k}$

$$\frac{\text{Sin}\,\alpha\,\text{Cos}\,\lambda - \text{Cos}\,\alpha\,\text{Sin}\,\lambda}{\text{Sin}\,\alpha} = \frac{1}{k}$$

$$\frac{\text{Sin}\,\alpha\,\text{Cos}\,\lambda}{\text{Sin}\,\alpha} - \frac{\text{Cos}\,\alpha\text{Sin}\,\lambda}{\text{Sin}\,\alpha} = \frac{1}{k}$$

If $\text{Cos}\;\lambda - \text{Cot}\,\alpha\,\text{Sin}\,\lambda = \frac{1}{k}$

If $\sin\;\lambda\left[\frac{\text{Cos}\,\lambda}{\text{Sin}\,\lambda} - \text{Cot}\,\alpha\right] = \frac{1}{k}$

If $k(\cot\lambda - \cot\alpha) = \operatorname{cosec}\lambda$ Which in required

Exercise:

1) a) A projectile is thrown with a velocity of 20m/sec at an elevation 30^0. Find the greatest height attained and horizontal range. b) A particle is projected with a velocity of 9.6 at an angle of 30^0. Find

i) The time of flight.

ii) The greatest height of the particle.

2) If the time of flight of a shot in T seconds over a range of x metres, show that the elevation is $\tan^{-1}\left(\frac{gT^2}{2x}\right)$ and determine the maximum height and the velocity of projection.

3) A body is projected at angle α to the horizontal, so as to clear two walls of equal height 'a' a distance 2a from each other. Show that the range is $2a\cot\frac{\alpha}{2}$.

6.4 Horizontal Range of projection:

To determine when the horizontal range of a projectile is maximum, given the magnitude u of the velocity of projection.

If u is the initial velocity and α is the angle of projection, the range R on the horizontal plane through the point of projection is given by

$$R = \frac{2u^2 \sin\alpha \cos\alpha}{g} = \frac{u^2 \sin 2\alpha}{g} \qquad (1)$$

Now g being a constant, for a given value of u, the value of R is greatest when $\sin 2\alpha$ is greatest

When $\sin 2\alpha = 1$

This happens when $2\alpha = 90^0$

(ie) $\alpha = 45^0$

Hence a maximum when the particle is projected at an angle of 45^0 to the horizontal (ie) the direction of projection for maximum horizontal range bisects the angle between the horizontal and the vertical. Also when $\alpha = 45^0$ from (1) $R = \frac{u^2}{g}$.

(ie) The maximum horizontal range is $\frac{u^2}{g}$.

Space for Hints

6.3.2 To show that for a given initial of projection, there are in general two possible directions of projections so as to obtain a given horizontal range

Let u be the velocity of projection of a particle, and α the necessary angle of projection so as to get a given horizontal range equal to k.

$$\text{Then } \frac{k}{1} = \frac{U^2 \operatorname{Sin} 2\alpha}{g}$$

$$\therefore \operatorname{Sin} 2\alpha = \frac{gk}{u^2} \qquad (1)$$

Since u and k are given and g is a constant, the R.H.S of (1) is a known positive quantity. If $gk<u^2$.

We can determine an acute angle θ whose since is exactly equal to $\frac{gk}{u^2}$

Then (1) becomes $\sin 2\alpha = \operatorname{Sin}\theta$ (2)

$$\therefore 2\alpha = \theta \text{ or } \alpha = \frac{\theta}{2}$$

Since $\sin(180^0 - \theta) = \sin\theta$, (2) can also be written as

$$\operatorname{Sin} 2\alpha = \operatorname{Sin}(180^0 - \theta)$$

$$2\alpha = 180^0 - \theta$$

$$\alpha = 90^0 - \theta/2 \qquad (4)$$

From (3) and (4), we find that there are two values of α and so two directions of projection, each giving the same range k.

Let α_1 and α_2 be these two values of α.

$$\text{Then } \alpha_1 = \frac{\theta}{2} \text{ and } \alpha_2 = 90^0 - \frac{\theta}{2}.$$

$$\therefore \alpha_1 + \alpha_2 = 90^0$$

As $\theta < 90^0, \alpha_1 < 45^0$ and so $\alpha_2 > 45^0$

$$\text{Now } 45^0 - \alpha_1 = 45^0 - \frac{\theta}{2}.$$

$$\text{And } \alpha_2 - 45^0 = 90^0 - \frac{\theta}{2} - 45^0 = 45^0 - \frac{\theta}{2}$$

$$\text{(ie) } 45^0 - \alpha_1 = \alpha_2 - 45^0 \qquad (5)$$

But 45^0 is the angle of projection to get maximum horizontal range will the same initial velocity. So (5) show that the two directions α_1 & α_2 are equally inclined to the direction of maximum range. This is shown in fig (a)

In fig (a) OT_1 and OT_2 are the directions α_1 & α_2 necessary to get a given range k. OT is the direction giving maximum horizontal range.

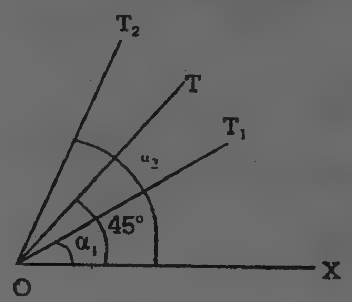


Fig. (a)

$\angle T_1OT = \angle XOT - \angle XOT_1$

$= 45^0 - \alpha_1$

$\angle T_2 OT = \angle XOT_2 - \angle XOT$

$= \alpha_2 - 45^0$

And $\angle T_1 OT = \angle T_2 OT$

In other words, OT bisects the angle between OT_1 and OT_2.

If $u^2 = g\,k$, from (1), $\sin 2\alpha = 1$

Then $2\alpha = 90^0$ or $\alpha = 45^0$.

Only one value of α is possible and this corresponds to the case of maximum range.

If $u^2 < gk$, the R.H.S of (1) is greater than 1 and so we cannot get a real value for α.

(ie) There is no angle of projection to get a range greater than $\frac{u^2}{g}$, which is really the maximum range possible.

Note:

To get a given horizontal range k, we find $u^2 > gk$. So the minimum value of $u = \sqrt{gk}$.

Example: 1

If h and h' be the greatest heights in the two paths of a projectile with a given velocity for a given range R, prove that $= 4\sqrt{(hh')}$

Space for Hints

Solution:

Let α and α' be the two angles of projection with a given velocity u to get a given range R.

Then we know that $\alpha+\alpha'=90^0$

(ie) $\alpha' = 90^0 - \alpha$ (1)

$$\text{Also } R = \frac{2u^2 \operatorname{Sin}\alpha \operatorname{Cos}\alpha}{g} \quad (2)$$

$$h = \frac{u^2 \operatorname{Sin}^2\alpha}{2g} \quad (3)$$

$$h' = \frac{u^2 \operatorname{Sin}^2 \alpha'}{2g} \quad (4)$$

$$\text{Hence } \sqrt{hh'} = \frac{u^2 \operatorname{Sin}\alpha \operatorname{Sin}\alpha'}{2g} = \frac{u^2 \operatorname{Sin}\alpha \operatorname{Sin}(90^0-\alpha)}{2g} \text{ using (1)}$$

$$= \frac{u^2 \operatorname{Sin}\alpha \operatorname{Cos}\alpha}{2g} = \frac{R}{4} \text{ using (2)}$$

(ie) $R = 4\sqrt{(hh')}$

Example: 2

A shell bursts on contact with the ground and pieces from it fly in all directions with all velocities upto 30metres per second. Show that a man 30m away is in danger for 5 secs nearly.

Solution:

Here the given velocity u = 30m, and the given range R = 30m

$$\text{Since } R = \frac{u^2 \operatorname{Sin}2\alpha}{g},$$

We have

$$\operatorname{Sin} 2\alpha = \frac{gR}{u^2} = \frac{9.81 \times 30}{30 \times 30} = 0.327$$

From the tables,

$2\alpha = 19^0\ 5'$ or $160^0\ 55'$

(ie) $\alpha = 9^0\ 32\frac{1}{2}'$ or $80^0\ 27\frac{1}{2}'$

These are the two angles of projection to get the given range.

Let t_1 and t_2 be the two times of flight for the two particular place which fly in the directions $9^0\ 32\frac{1}{2}'$ and $80^0\ 27\frac{1}{2}'$ respectively.

These two places will strike the man and so he is in danger for the interval $t_2 - t_1$ secs.

$$\text{Now, } t_1 = \frac{2u \operatorname{Sin} 9^0\ 32\frac{1}{2}'}{g} = \frac{2 \times 30 \sin 9^0\ 32\frac{1}{2}'}{9.8} = 1.014$$

$$T_2 = \frac{2u \operatorname{Sin} 80^0\ 27\frac{1}{2}'}{g} = \frac{2 \times 30 \operatorname{Sin} 80^0\ 27\frac{1}{2}'}{9.81} = 6.032$$

Period of danger $= t_2 - t_1$

$$= 6.032 - 21.014 = 5.018$$

$$= 5 \text{ secs (nearly)}$$

Example: 3

The range of a rifle bullet is 1000m, when α is the angle of projection. Show that if the bullet is fired with the same elevation from a car traveling 36km/n towards the target, the range will be increased by

$$\frac{1000\sqrt{\tan\alpha}}{7} \text{m} \cdot (g = 9.8 \text{ m/sec}^2)$$

Solution:

Let u m/sec be the velocity of projection.

The horizontal range

$$R = \frac{2u^2 \operatorname{Sin}\alpha \operatorname{Cos}\alpha}{g} = 1000 \text{(given)} \qquad (1)$$

$$\text{Also } R = \frac{2}{g}(u \operatorname{Cos}\alpha).(u \operatorname{Sin}\alpha)$$

$$= \frac{2}{g} \text{ (horizontal velocity) x (initial vertical velocity)} \qquad (2)$$

When the bullet is fired from a moving car, the horizontal velocity is increased and the increases

$$= 36 \text{ km/h} = \frac{36 \times 1000}{60 \times 60} = 10 \text{m/sec}.$$

Now horizontal velocity $= u \cos\ \alpha + 10$

Space for Hints

As there is no change in the vertical motion, new initial vertical velocity $= u\,\mathrm{Sin}\,\alpha$.

Hence in the second case, horizontal range

$R' = \frac{2}{g}(u\,\mathrm{Cos}\,\alpha + 10)(u\,\mathrm{Sin}\,\alpha)$ using the form given in (2)

$$R' - R = \frac{2}{g}(u\,\mathrm{Cos}\,\alpha + 10)(u\,\mathrm{Sin}\,\alpha) - \frac{2u^2\,\mathrm{Sin}\,\alpha\,\mathrm{Cos}\,\alpha}{g}$$

$$= \frac{20u\,\mathrm{Sin}\,\alpha}{g} \qquad (3)$$

From (1)

$$u^2 = \frac{g \times 1000}{2\,\mathrm{Sin}\,\alpha\,\mathrm{Cos}\,\alpha} = \frac{500g}{\sin\alpha\,\mathrm{Cos}\,\alpha}$$

Putting this value of u in (3)

$$R' - R = \frac{20\mathrm{Sin}\,\alpha}{g} \times \sqrt{\frac{500g}{\mathrm{Sin}\,\alpha\cos\alpha}}$$

$$= \frac{20 \times 10 \times \sqrt{\tan\alpha \times 5}}{\sqrt{g}}$$

$$= \frac{200\sqrt{\tan\alpha}\,\sqrt{5}}{\sqrt{9.8}} = \frac{200\sqrt{\tan\alpha}\,\sqrt{5} \times \sqrt{10}}{\sqrt{98}}$$

$$= \frac{200\sqrt{\tan\alpha}\,\sqrt{5} \times \sqrt{5 \times}\,\sqrt{2}}{7\sqrt{2}}$$

$$= \frac{1000\sqrt{\tan\alpha}}{7}\,\mathrm{m}.$$

And this is the increase in the range.

Example: 4

A body is projected with the same velocity at two angles cover the same horizontal range R. If T_1 and T_2 be the two times of flight, prove that R $= \frac{1}{2}gT_1T_2$.

Solution:

We know that, with a given velocity of projection, u

We can obtain a given horizontal range R, with two angles of projection, α_1 or $90 - \alpha_1$, For the two paths, the times of flight are given by,

Space for Hints

$$T_1 = \frac{2u \sin \alpha_1}{g}$$

And $T_2 = \dfrac{2u \sin (90^0 - \alpha_1)}{g} = \dfrac{2u \sin\alpha_1}{g}$

$$\therefore \frac{1}{2} g T_1 T_2 = \frac{1}{2} g. \frac{2u\sin\alpha_1}{g}, \frac{2u\sin \alpha_1}{g}$$

$$= \frac{u^2 \sin 2\alpha_1}{g}$$

$$= R.$$

$$\therefore R = \frac{1}{2} T_1 T_2.$$

Example: 5

A particle is projected so as to clear a wall of height h at a horizontal distance a and to have a range b from the point of projection. Find the velocity of projection.

Solution:

The equation to the projectile is $y = x \tan \alpha - \dfrac{gx^2}{2u^2 \cos^2 \alpha}$

This passes through (a,h)

$$h = a \tan \alpha - \frac{ga^2}{2u^2 \cos^2 \alpha} \qquad (1)$$

$$\text{Range} = b = \frac{u^2 \sin 2a}{8} \qquad (2)$$

i,e., $\dfrac{u^2}{8} = \dfrac{b}{\sin 2\alpha}$

Substituting in (1)

$$h = a \tan \alpha - \frac{a^2}{2\cos^2\alpha} \frac{\sin 2\alpha}{b}$$

$$= a \tan \alpha - \frac{a^2}{b} \tan \alpha$$

$$= a \tan \alpha \left(\frac{b-a}{b} \right)$$

$$\tan \alpha = \frac{hb}{b-a} \qquad (3)$$

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From (2)

$$u^2 = \frac{gb}{2\tan\alpha}(1+\tan^2\alpha)$$

Substituting for tan α from (3)

$$u^2 = \frac{gb}{\frac{2bh}{a(b-a)}}\left[1+\frac{h^2 b^2}{a^2(a-b)^2}\right]$$

$$= \frac{gb}{2bha(b-a)}\left[a^2(b-a)^2 + h^2 b^2\right]$$

$$u = \sqrt{\frac{g\left[a^2(b-a)^2 + h^2 b^2\right]}{2ha(b-a)}}$$

Example: 6

Prove the relation $gT^2 = 2R\tan\alpha$, where T is the time of flight and R is the horizontal range, α the angle of projection. If $\alpha = 60^0$, find in terms of R the height of the projectile, when it has moved through a distance equal to $\frac{3R}{4}$.

Solution:

$$T = \frac{2u\,\mathrm{Sin}\,\alpha}{g} \qquad \text{................ (1)}$$

$$\text{and } R = \frac{u^2\,\mathrm{Sin}2\alpha}{g^2} \qquad \text{.................(2)}$$

$$gT^2 = g\frac{4u^2\,\mathrm{Sin}^2\alpha}{g^2}$$

$$= \frac{4g\,\mathrm{Sin}^2\alpha}{g^2}\,\frac{gR}{\mathrm{Sin}2\alpha} \text{ using (2)}$$

$gT^2 = 2R\tan\alpha$

The equation to the path of the projectile is

$$y = x\tan\alpha - \frac{gx^2}{2u^2\,\mathrm{Cos}^2\alpha}$$

Putting $\alpha = 60^0, x = \frac{3R}{4}$

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and $u^2 = \frac{gR}{\text{Sin}\,120^0} = \frac{2gR}{\sqrt{3}}$

$$\therefore y = \frac{3R}{4}\sqrt{3} - \frac{g9R^2}{\dfrac{2 \times 16 \times gR \times 2}{\sqrt{3} \times 4}}$$

$$= \frac{3R\sqrt{3}}{4} - \frac{9\sqrt{3}R}{16}$$

$$y = \frac{3\sqrt{3R}}{16}.$$

$$y = \frac{3\sqrt{3}R}{16}.$$

Example: 7

Two masses, projected from the same point with the same velocity at elevations of α and β to the horizontal are aimed at a target on the horizontal plane through the point of projection. The first one falls x metres too short of the target while the second one goes y metres beyond the target. If θ is the elvation to hit the target correctly with the same velocity of projection, Show that $(x + y)\sin 2\theta = x\sin 2\beta + y\sin 2\alpha$.

Solution:

Let u be the velocity of projection for each case. Let R be the horizontal range of the target. Then as per the condition of the problem we have

$$R - x = \frac{u^2\,\text{Sin}\,2\alpha}{g} \qquad (1)$$

$$R + y = \frac{u^2\,\text{Sin}\,2\beta}{g} \qquad (2)$$

$$R = \frac{u^2\,\text{Sin}\,2\theta}{g} \qquad (3)$$

Substituting for R from (3) in (1) and (2) and simplifying we get

$$\frac{u^2}{g}(\text{Sin}2\theta - \sin 2\alpha) = x$$

$$\frac{u^2}{g}(\text{Sin}2\beta - \sin 2\theta) = y$$

Dividing the equation

$$\frac{\text{Sin }2\theta - \text{Sin }2\alpha}{\text{Sin }2\beta - \text{Sin }2\theta} = \frac{x}{y}$$

$$y \sin 2\theta - y \sin 2\alpha = x \sin 2\beta - x \sin 2\theta$$

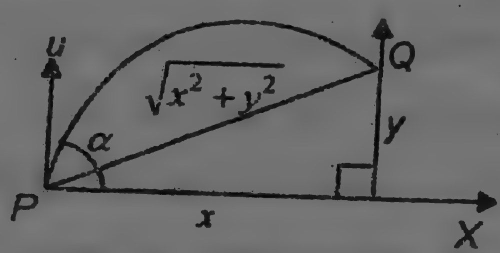
$$(x + y) \sin 2\theta = x \sin 2\beta + y \sin 2\alpha.$$

Example: 8

A particle is to be projected from a point P so as to pass through another point Q. Show that the product of the two times of flight of flight from P to Q with a given velocity of projection is $\frac{2PQ}{8}$.

Solution:

Let u be the velocity and α the angle of projection. If the coordinates of Q with reference to the rectangular axes through P are (x, y) then



$$x = u \cos \alpha t \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$y = u \sin \alpha t - \frac{1}{2} g t^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Eliminating α from (1), (2)

We get

$$(u \cos \alpha t)^2 + (u \text{ Sin } \alpha t)^2$$

$$= x^2 + \left(y + \frac{1}{2} g t^2\right)^2$$

$$\therefore x^2 + y^2 + g t^2 y + \frac{g^2 t^4}{4} = u^2 t^2$$

i.e., $\frac{1}{4} g^2 t^4 + t^2 (yg - u^2) + (x^2 + y^2) = 0$

This is a quadratic in t^2 has two roots t_1^2 and t_2^2 so that

$$t_1^2 \, t_2^2 = \frac{x^2 + y^2}{\frac{1}{4} g^2}$$

$$t_1^2\, t_2^2 = \frac{2\sqrt{x^2+y^2}}{8} = \frac{2PQ}{g}.$$

Example: 9

If v_1 and v_2 be the velocities at the ends of a focal chord of the projectile's path and v horizontal component of the velocity, show that $\frac{1}{v_1^2}+\frac{1}{v_2^2}=\frac{1}{v^2}$.

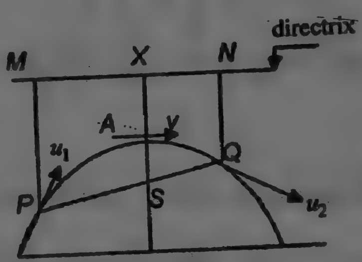
Solution:

Let PSQ be a focal chord; the velocities at P and Q be v_1 and v_2 respectively. The velocity at the vertex is v which is the horizontal component

By a known result, we have

$$v_1^2 = 2g\,PM = 2g\,SP$$

$$v_2^2 = 2g\,QN = 2g\,SQ$$



$$\therefore \frac{1}{v_1^2}+\frac{1}{v_2^2}=\frac{1}{2g}\left(\frac{1}{SP}+\frac{1}{SQ}\right)$$

But in a parabola $\frac{1}{SP}+\frac{1}{SQ}=\frac{2}{\ell}$ where ℓ is the semi – latusrectum of the parabola.

$$\therefore \frac{1}{v_1^2}+\frac{1}{v_2^2}=\frac{1}{2g}\frac{2}{\ell}$$

$$= \frac{1}{g2SA}$$

$$= \frac{1}{2gAX}=\frac{1}{v^2}.$$

Example: 10

A particle is projected with a velocity $2\sqrt{ag}$ so that just clears two walls of equal height a which are at distance 2a from each other. Show that the latus

Space for Hints

recturm of the path is equal to 2a and the time of passing between the wall is $2\sqrt{\frac{a}{g}}$.

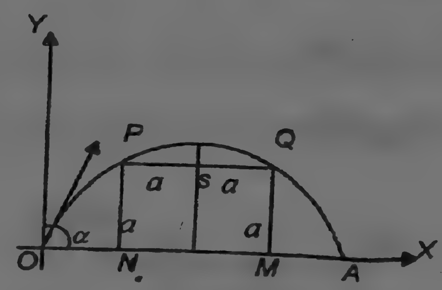
Solution:

Height of the directrix above O

$$= \frac{u^2}{2g} = \frac{4ag}{2g} = 2a$$

Height of the directrix above P

$$= 2a - a = a$$



We know that the distances of a point on the parabola from directrix and focus are equal. Hence if S be the focus, PS = a ∴ S lies an PQ. Hence PQ = latusrectum = 2a.

Also latusrectum

$$= \frac{2u^2 \text{Cos}^2}{g} = 2a$$

$$\therefore u\cos\alpha = \sqrt{ag}$$

If t be the time from P to Q

$$u\cos\alpha t = PQ = 2a$$

$$\therefore t = \frac{2a}{u\cos\alpha}$$

$$= \frac{2a}{\sqrt{ag}} = 2\sqrt{\frac{a}{g}}.$$

Example: 11

P is a point at a horizontal distance a and vertical distance b from the point of projection. Show that in order that the particle may reach P,

$u^2 \nless g(b+\sqrt{a^2+b^2})$. For any u satisfying this inequality, show that there are two possible directions of projection in order that a particle may reach P.

Solution:

The equation to the path of the projectile is

$$y = x \tan \alpha - \frac{gx^2}{2u^2 \operatorname{Cos}^2 \alpha}$$

The point (a, b) lies on this path.

$$\therefore b = at - \frac{ga^2}{2u^2}(1+t^2) \text{ where } t = \tan \alpha$$

$$\text{i.e., } \frac{ga^2}{2u^2}t^2 - at + b + \frac{ga^2}{2u^2} = 0 \qquad (1)$$

This is a quadratic in t.

In order that the value of t may be real, we should have

$$B^2 - 4AC \nless 0$$

$$a^2 - \frac{4ga^2}{2u^2}\left(b + \frac{ga}{2u^2}\right) \nless 0$$

$$\text{I.e., } a^2u^2 - 2ga^2b - \frac{2g^2a^4}{2u^2} \nless 0 \qquad (2)$$

$$\frac{a^2}{u^2}(u^4 - 2gbu^2 - g^2a^2) \nless 0$$

This expression inside the bracket is a quadratic in u^2 with the u^4 with the u^4 terms coefficient positive; and the expression should not be negative and hence u^2 does not lie between the roots of $u^4 - 2gbu^2 - g^2a^2 = 0$

$$u^2 = \frac{2gb \pm \sqrt{4g^2b^2 + 4g^2a^2}}{2}$$

$$= g(b \pm \sqrt{b^2 + a^2})$$

u^2 being a perfect square cannot be less than the negative root. Hence it must be greater than or equal to the positive root.

$$\therefore u^2 \geq g\left(b + \sqrt{a^2+b^2}\right)$$

$$\text{i.e., } u^2 \nless g\left(b + \sqrt{a^2+b^2}\right)$$

When this inequality is satisfied, condition (2) is also satisfied and hence equation (1) has two real values of t corresponding to each of which there is a direction of projection.

Example: 12

If the time of flight of a shot is T seconds over a range of x meters, show that the elevation is $\tan^{-1}\left(\frac{gT^2}{2x}\right)$ and determine the maximum height and the velocity of projection.

Solution:

Let α be the angle of projection.

Let T = Time of flight

$$= \frac{2u\text{Sin}\alpha}{g} \qquad (\because \text{by definition})$$

$$x = \text{Range} = \frac{u^2\,\text{Sin}2\alpha}{g}$$

Now $$\frac{gT^2}{2x} = g.\frac{(2u\sin\alpha)^2}{g^2}\Big/2\ \frac{u^2\,\text{Sin}2\alpha}{g}$$

$$= \frac{4u^2\,\text{Sin}^2\alpha}{g} \text{ x } \frac{g}{2.u^2.2\,\text{Sin}\,\alpha\,\text{Cos}\,\alpha}$$

$$= \frac{\text{Sin}\,\alpha}{\text{Cos}\,\alpha} \qquad \Rightarrow \tan\alpha = \frac{gT^2}{2x} \qquad \text{(A)}$$

$$\alpha = \tan^{-1}\left(\frac{gT^2}{2x}\right)$$

Maximum height $$h = \frac{u^2\,\text{Sin}^2\alpha}{2g} \qquad (1)$$

Time $$T = \frac{2u\,\text{Sin}\,\alpha}{g}$$

$$\Rightarrow \text{Sin}\,\alpha = \frac{Tg}{2u} \qquad (2)$$

From (1) and (2) we get

Maximum height $$h = u^2\left(\frac{Tg}{2u}\right)^2\Big/2g$$

Space for Hints

$$= u^2 \frac{T^2 g^2}{4u^2} / 2g$$

$$h = T^2 g/8$$

To find the velocity of projection:

$$\text{Range } x = \frac{u^2 \sin 2\alpha}{g}; \text{Time } T = \frac{2u \sin\alpha}{g}$$

Now $$\frac{x}{T} = \left(\frac{u^2 \sin 2\alpha}{g}\right) \Big/ \left(\frac{2u \sin\alpha}{g}\right)$$

$$= \frac{u^2\, 2\sin\alpha\cos\alpha}{g} \times \frac{g}{2u\sin\alpha} = u\cos\alpha$$

$$\therefore \frac{x}{T} = u\cos\alpha \qquad \Rightarrow u = \frac{x}{T\cos\alpha}$$

$$= \frac{x}{T}\sec\alpha$$

$$= \frac{x}{T}\sqrt{1+\tan^2\alpha}$$

$$= \frac{x}{T}\sqrt{1+\left(\frac{gT^2}{2x}\right)^2} \quad (\therefore \text{by} (A))$$

$$= \frac{x}{T}\sqrt{1+\frac{g^2T^4}{4x^2}}$$

$$= \frac{x}{T}\sqrt{(4x^2+g^2T^4)/4x^2}$$

$$u = \frac{x}{T}\frac{\sqrt{4x^2+g^2T^4}}{2x}$$

$$\therefore u = \frac{\sqrt{4x^2+g^2T^4}}{2T}$$

Exercise:

1) If R be the horizontal range of a projectile and h its greatest height, prove that the maximum horizontal range with the same velocity of projection in $\frac{16h^2+R^2}{8h}$

Space for Hints

2) Prove that in any trajectory over a horizontal plane, the horizontal range is a maximum when it is equal to four times the greatest height.

3) A shot projected with velocity v at an elevation of 45^0 reaches a point P on the horizontal plane through the point of projection. Show that in order to hit a mark is h feet above p, if the shot is projected at the same elevation, the velocity of projection must be, increased to $\dfrac{v^2}{\left(v^2 - gh\right)^{1/2}}$

UNIT – 7

RANGE ON AN INCLINED PLANE

7.1 Theorems and problems

Theorem (1): From a point on a plane, which is inclined at an angle β with the horizon a particle is projected with a velocity u at an angle α with the horizontal in a plane passing through the normal to the inclined plane and the line of greatest slope. To find the range on the inclined plane.

Proof:

Let P be the point of projection and the particle strike the inclined plane at Q. Then PQ is the range on the inclined plane.

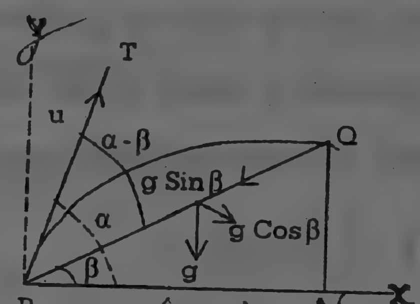


fig. (7.1)

Let PQ = r. Taking P as the origin and the horizontal and the vertical through P as the axes of x and y respectively. The equation to the path is,

$$Y = x \tan \alpha - \frac{gx^2}{2u^2 \operatorname{Cos}^2 \alpha} \qquad (1)$$

Draw QN $\perp^r$ to the horizontal plane through P. The coordinates of Q are $(r \operatorname{Cos}\beta, r \sin\beta)$.

Substituting these in (1)

$$r \sin \beta - r \operatorname{Cos}\beta . \tan \alpha - \frac{gr^2 \operatorname{Cos}^2 \beta}{2u^2 \operatorname{Cos}^2 \alpha}.$$

Multiplying by $2u^2 \cos^2 \alpha$ and canceling r throughout,

We have,

$$2u^2 \operatorname{Cos}^2 \alpha \ S \text{ in } \beta = 2u^2 \operatorname{Cos}\beta \operatorname{Sin}\alpha \operatorname{Cos}\alpha - gr \operatorname{Cos}^2 \beta.$$

$$\therefore r = \frac{2u^2 \operatorname{Cos}\beta \operatorname{Sin}\alpha \operatorname{Cos}\alpha - 2u^2 \operatorname{Cos}^2 \alpha \operatorname{Sin}\beta}{g \operatorname{Cos}^2 \beta}$$

$$= \frac{2u^2 \operatorname{Cos}\alpha (\operatorname{Sin}\alpha \operatorname{Cos}\beta - \operatorname{Cos}\alpha \operatorname{Sin}\beta)}{g \operatorname{Cos}^2 \beta}$$

$$\text{(ie) } r = \frac{2u^2 \operatorname{Cos}\alpha \operatorname{Sin}(\alpha-\beta)}{g \operatorname{Cos}^2\beta}$$

Aliter:

We can study separately the motion of the particle along the inclined plane and the motion $\perp^r$ to the plane. The initial velocity u can be resolved into two components i) u Cos $(\alpha-\beta)$, along PQ, the inclined plane $\perp^r$ and (ii) $u\sin(\alpha-\beta)$ to the inclined plane.

The acceleration g can be resolved into two components i) g Cos $\beta \perp^r$ to the inclined plane in the downward direction and ii) g sin β along the inclined plane towards P. This resolution is shown in the figure.

Let T be the time which the particle takes to go from P to Q.

After time T, the particle is again on the inclined plane and so, during time T, the distance traveled $\perp^r$ to the inclined plane is = o.

$$\therefore O = u \operatorname{Sin}(\alpha-\beta) T - \frac{1}{2} g \operatorname{Cos}\beta . T^2$$

$$\text{(ie)} \qquad T = \frac{2u \operatorname{Sin}(\alpha-\beta)}{g \operatorname{Cos}\beta}$$

This is the time of flight on the inclined plane. During this time, the horizontal velocity remains constant and u cos α. So, horizontal distance described in time T = PN = u cos αT.

But PN = PQ Cos β.

$$PQ \operatorname{Cos}\beta = u \operatorname{Cos}\alpha T.$$

$$\text{(ie) } PQ = \frac{u \operatorname{Cos}\alpha}{\operatorname{Cos}\beta} . T$$

$$= \frac{u \operatorname{Cos}\alpha}{\operatorname{Cos}\beta} . \frac{2u \operatorname{Sin}(\alpha-\beta)}{g \operatorname{Cos}\beta}$$

$$PQ = \frac{2u^2 \operatorname{Sin}(\alpha-\beta) \operatorname{Cos}\alpha}{g \operatorname{Cos}^2\beta}$$

Theorem: (2)

To find the greatest distance of the projectile from the inclined plane and show that it is attained in half the total time of flight.

Proof:

Let us consider the motion $\perp^r$ to the inclined plane. As explained in range on an inclined plane. The initial velocity in this direction is u sin $(\alpha-\beta)$ and this is subject to an acceleration g cos β is the same direction but acting down wards. Let y be the distance travelled by the particle in this direction in time t.

Then

$$y = u \sin(\alpha-\beta).\, t - \tfrac{1}{2} g \cos \beta . t^2 \qquad (1)$$

Differentiating with respect to t.

$$\frac{dy}{dt} = u \operatorname{Sin}(\alpha-\beta) - g \operatorname{Cos}\beta . t$$

$$\text{And } \frac{dy^2}{dt^2} = -g \operatorname{Cos}\beta = \text{negative.}$$

So, y is maximum, when $\frac{dy}{dt} = O$.

(ie) when u sin $(\alpha-\beta)$ - g Cos β.t=O

$$\text{(ie) } t = \frac{u \operatorname{Sin}(\alpha-\beta)}{g \operatorname{Cos}\beta} \qquad (3)$$

Substituting (3) in (1) maximum value of y

$$= u \sin(\alpha-\beta).\, \frac{u \operatorname{Sin}(\alpha-\beta)}{g \operatorname{Cos}\beta} - \frac{1}{2} g \operatorname{Cos}\beta \frac{u^2 \operatorname{Sin}^2(\alpha-\beta)}{g^2 \operatorname{Cos}^2\beta}$$

$$= \frac{u^2 \operatorname{Sin}^2(\alpha-\beta)}{g \operatorname{Cos}\beta} - \frac{u^2 \operatorname{Sin}^2(\alpha-\beta)}{2g \operatorname{Cos}\beta}$$

$$= \frac{u^2 \operatorname{Sin}^2(\alpha-\beta)}{2g \operatorname{Cos}\beta} \qquad (4)$$

(4) is the greatest distance of the projectile from the inclined plane. Also. From (3), time to this greatest distance.

$= \frac{u \operatorname{Sin}(\alpha-\beta)}{g \operatorname{Cos}\beta}$ and this is clearly half of the time of flight.

Aliter:

When the particle is at the greatest distance from the inclined plane, it will have all its velocity only parallel to the inclined plane. Hence the component velocity $\perp^r$ to the inclined plane is zero. So, if S is the greatest distance

We have $O = [u \operatorname{Sin}(\alpha-\beta)]^2 - 2g \operatorname{Cos}\beta, S$

$$\text{(ie) } S = \frac{u^2 \sin^2(\alpha-\beta)}{2g\cos\beta}$$

Also if t is the corresponding time,

$$O = u\sin(\alpha-\beta) - g\cos\beta t$$

$$\text{(Or) } t = \frac{u\sin(\alpha-\beta)}{g\cos\beta}$$

Theorem: (3)

To determine when the range on the inclined plane is maximum, given the magnitude u of the velocity of projection:-

Proof:

From the Range of inclined plane

The range R on the inclined plane is given by

$$R = \frac{2u^2 \cos\alpha \sin(\alpha-\beta)}{g\cos^2\beta}$$

$$= \frac{u^2}{g\cos^2\beta}\left[\sin(2\alpha-\beta)-\sin\beta\right] \qquad (1)$$

Now u and β are given.

The quantity outside the bracket, $\frac{u^2}{g\cos^2\beta}$ is constant.

So R is maximum, when the value of the expression inside the bracket is a maximum

(ie) when $\sin(2\alpha-\beta)$ is greatest.

(ie) when $2\alpha-\beta=\frac{\pi}{2}$.

(ie) $\alpha=\frac{\pi}{4}+\frac{\beta}{2}$ for maximum range.

When α takes this value, $\alpha-\beta=(2\alpha-\beta)-\alpha$

$$= 90^0 - \alpha \qquad (2)$$

Referring to fig (7.1)

$$\alpha-\beta=\angle TPN-\angle QPN=\angle TPQ$$

And $90^0 - \alpha = \angle YPT$

Hence from (2), $\angle TPQ=\angle YPT$

(ie) PT, the direction of projection for maximum range bisects the angle between the vertical and the inclined plane.

From (1), the value of maximum range

=

$$\frac{u^2}{g\cos^2\beta}(1-\sin\beta)=\frac{u^2}{g(1-\sin^2\beta)}(1-\sin\beta)=\frac{u^2(1-\sin\beta)}{g(1+\sin\beta)(1-\sin\beta)}=\frac{u^2}{g(1+\sin\beta)}$$

Theorem: (4)

To show that, for a given initial velocity of projection, there are in general two possible directions of projection so as to obtain a given range on an inclined plane:

Proof:

Let u be the velocity of projection of a particle and α the necessary angle of projection so as to get a given range k on an inclined plane of inclined plane of inclination β to the horizontal.

$$\text{Then } k=\frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta} \qquad \frac{u^2(\sin(2\alpha-\beta)-\sin\beta)}{g\cos^2\beta} \tag{1}$$

$$\text{From (1) } \sin(2\alpha-\beta)=\frac{gk\cos^2\beta}{u^2}+\sin\beta \tag{2}$$

Since k,u,β are given the R.H.S of (2) is a know positive quantity. So we can determine an acute angel θ Whose sine is exactly to $\frac{gk\cos^2\beta}{u^2}+\sin\beta$.

$$\text{Then (2) becomes, } \sin(2\alpha-\beta)=\sin\theta \tag{3}$$

$$\text{(ie) } 2\alpha-\beta=\theta \text{ or } \alpha=\frac{\theta}{2}+\frac{\beta}{2} \tag{4}$$

$$\text{(ie) } \alpha=90^0-\frac{\theta}{2}+\frac{\beta}{2} \qquad [\because 2\alpha-\beta=180^0-\theta] \tag{5}$$

From (4) and (5)

We find that there are two value of α and so two directions of projection, each giving the same range k.

Let α_1 and α_2 these two values of α.

$$\text{Then } \alpha_1=\frac{\theta}{2}+\frac{\beta}{2} \text{ and } \alpha_2=90^0-\frac{\theta}{2}+\frac{\beta}{2}$$

Space for Hints

Now $(45^0 + \frac{\beta}{2}) - \alpha_1 = 45^0 + \frac{\beta}{2} - \frac{\theta}{2} - \frac{\beta}{2} = 45^0 - \frac{\theta}{2}$

And $\alpha_2 - (45^0 + \frac{\beta}{2}) = 90^0 - \frac{\theta}{2} + \frac{\beta}{2} - 45^0 - \frac{\beta}{2} = 45^0 - \frac{\theta}{2}$

$$\therefore \left(45^0 + \frac{\beta}{2}\right) - \alpha_1 = \alpha_2 - \left(45^0 + \frac{\beta}{2}\right) \qquad (6)$$

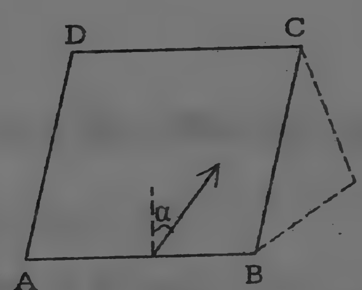
But $45^0 + \frac{\beta}{2}$ is the angle of projection for maximum range on the inclined plane. So (6) shows that the two directions α_1 and α_2 are equally inclined to the direction of maximum range.

Note:

The direction of projection is expressed as, an elevation to the horizontal. We can also take the elevation relative to the inclined plane. In problems, It should be carefully found out which of these angles is given.

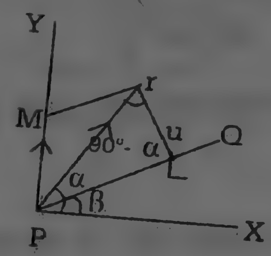
7.1.1 Motion on the surface of smooth inclined plane:

Let a particle be projected with velocity u on the surface of a smooth inclined plane ABCD of slope β in a direction inclined at an angle α to the line of greatest slope of the plane. The acceleration due to gravity can be resolved into two components, one, g sin β is the direction of the line of greatest slope and the other g Cos $\beta \perp^r$ to the inclined plane. But this component g cos β is opposed by an equal normal reaction of the inclined plane and so, the particle moves with an acceleration g sin β parallel to the line of greatest slope.



Example: 1

If U and V be the oblique components of the initial velocity in the vertical direction and is the direction of the line of greatest slope, show that the range on the inclined plane is $2\frac{UV}{g}$.



P is the point of projection and PX, PY are the horizontal and the vertical through P, PQ is the line of greatest slope of the inclined plane, with inclination β. PT is the initial direction of projection at an angle α to the horizontal. Let PT = the initial velocity u on some scale.

From T, draw TL|| PY and TM|| PQ as show is the figure.

The PM(=TL) and PL (= MT) are the oblique components of the initial velocity u in the vertical direction and in the direction of the line of greatest slope respectively.

From ΔPTL

We have

$$\frac{TL}{\text{Sin}\angle TPL}=\frac{PL}{\text{Sin}\angle PTL}=\frac{PT}{\text{Sin}\angle PLT} \qquad (1)$$

$$\angle TPL=\angle TPX=\angle LPX=\alpha-\beta$$

$$\angle PTL= \text{alternate } \angle MPT=90^0-\alpha$$

$$\angle PLT=180^0-(\angle PTL+\angle TPL)$$

$$= 180^0 - 90^0 - \alpha+\alpha-\beta)=90^0+\beta$$

Also PT = u. TL = U and PL = V.

Substituting in (1)

We have

$$\frac{u}{\text{Sin}\,(\alpha-\beta)}=\frac{V}{\text{Sin}\,(90^0-\alpha)}=\frac{u}{\text{Sin}(90^0+\beta)}$$

(ie) $$\frac{U}{\text{Sin}\,(\alpha-\beta)}=\frac{V}{\text{Cos}\,\alpha}=\frac{u}{\text{Cos}\,\beta}$$

$$\therefore u=\frac{u\text{Sin}\,(\alpha\_\beta)}{\text{Cos}\beta}\ \&\ V=\frac{u\,\text{Cos}\,\alpha}{\cos\beta}$$

$$2\frac{UV}{g}=\frac{2u\text{Sin}\,(\alpha-\beta)}{g\,\text{Cos}\beta}\times=\frac{u\,\text{Cos}\,\alpha}{\text{Cos}\,\beta}$$

Space for Hints

$$= \frac{2u^2 \operatorname{Sin}(\alpha-\beta)\operatorname{Cos}\alpha}{g \operatorname{Cos}^2 \beta}$$

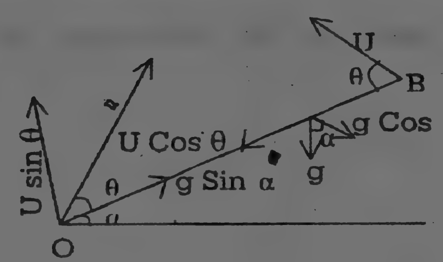
= range on the inclined plane.

Example: 2

Show that, for a given velocity of projection the maximum range down an inclined plane of inclination α bears to the maximum range up the inclined plane the ratio $\frac{1+\operatorname{Sin}\alpha}{1-\operatorname{Sin}\alpha}$.

Solution:

Let u be the given velocity of projection and θ the inclination of the direction of projection with the plane.



The velocity u can be resolved into two components u cos θ along the upward inclined plane and u sin $\theta \perp^r$ to the inclined plane. The acceleration g can be resolved into two components, g sin α along the downward inclined plane and g cos $\alpha \perp^r$ to the inclined plane and down wards.

Consider the motion perpendicular to the inclined plane.

Let T be the time of flight. Distance traveled $\perp^r$ to the inclined plane in Time T is = O.

$$\therefore O = u \operatorname{Sin}\theta . T - \frac{1}{2} g \operatorname{Cos}\alpha . T^2$$

$$\text{(ie) } T = \frac{2u \operatorname{Sin}\theta}{g \operatorname{Cos}\alpha}$$

During this time, the distance travelled along the plane

$$= u \operatorname{Cos} \theta . T - \frac{1}{2} g \operatorname{Cos}\alpha \cdot T^2$$

$$= u \operatorname{Cos} \theta . \frac{2u\operatorname{Sin}\theta}{g \operatorname{Cos}\theta} - \frac{1}{2} g \operatorname{Sin}\alpha . \frac{4u^2}{g^2} . \frac{\operatorname{Sin}^2\theta}{\operatorname{Cos}^2\alpha}$$

$$= \frac{2u^2 \operatorname{Sin}\theta \operatorname{Cos}\theta}{g^2 \operatorname{Cos}^2 \alpha} - \frac{2u^2 \operatorname{Sin}\alpha . \operatorname{Sin}^2 \theta}{g \operatorname{Cos}^2 \alpha}$$

Space for Hints

$$= \frac{2u^2 \operatorname{Sin}\theta}{g \operatorname{Cos}^2 \alpha}(\operatorname{Cos}\alpha \operatorname{Cos}\theta - \operatorname{Sin}\alpha \operatorname{Sin}\theta)$$

$$= \frac{2u^2 \operatorname{Sin}\theta}{g \operatorname{Cos}^2 \alpha}\operatorname{Cos}(\theta+\alpha)$$

$$= \frac{u^2}{g \operatorname{Cos}^2 \alpha}.2\operatorname{Cos}(\theta+\alpha)\operatorname{Sin}\theta$$

$$= \frac{u^2}{g \operatorname{Cos}^2 \alpha}[\operatorname{Sin}(20+\alpha) - \operatorname{Sin}\alpha)]$$

This is the rang R_1 up the inclined plane.

R_1 is maximum, when Sin $(20+\alpha)=1$

∴ Maximum range up the plane

$$= \frac{u^2}{g \operatorname{Cos}^2 \alpha}(1-\operatorname{Sin}\alpha) = \frac{u^2}{g(1+\operatorname{Sin}\alpha)}$$

When the particle is projected down the plane from B at the same angle to the plane, the time of flight has the same value $\frac{2u \sin\theta}{g \operatorname{Cos}\theta}$. But the component of the initial velocity along the inclined plane is u cos θ downwards and the component acceleration g sin α also downwards. Hence range down the plane.

R_2 = distance traveled along the plane in time T

$$= u \cos\theta.\ T + \frac{1}{2} g \operatorname{Sin}\alpha.\ T^2$$

$$= \frac{2u^2 \operatorname{Sin}\theta}{g \operatorname{Cos}^2 \alpha}(\operatorname{Cos}\alpha \operatorname{Cos}\theta + \operatorname{Sin}\alpha \operatorname{Sin}\theta)$$

$$= \frac{2u^2 \operatorname{Sin}\theta}{g \operatorname{Cos}^2 \alpha}\operatorname{Cos}(\theta-\alpha)$$

$$= \frac{u^2}{g \operatorname{Cos}^2 \alpha}[\operatorname{Sin}(2\theta-\alpha) - \operatorname{Sin}\alpha]$$

R^2 is maximum, when sin $(2\theta - \alpha) = 1$

So maximum range down the plane

$$= \frac{u^2}{g \operatorname{Cos}^2 \alpha}(1+\operatorname{Sin}\alpha) = \frac{u^2}{g(1-\operatorname{Sin}\alpha)}$$

$$\therefore \frac{\text{Max. range down the plane}}{\text{Max. range up the plane}} = \frac{u^2}{g(1-\text{Sin}\,\alpha)} \cdot \frac{g(1+\text{Sin}\,\alpha)}{u^2}$$

$$= \frac{1+\text{Sin}\,\alpha}{1-\text{Sin}\,\alpha}$$

Note:

The range R_2 down the plane can be got from the range R_1 up the plane by changing α into - α.

Example: 3

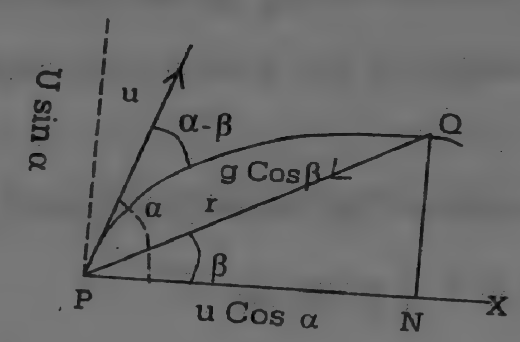
A particle is projected at an angle α with a velocity u and it strike up an inclined plane of inclination β at right angles to the plane. Prove that

i) $\text{Cot}\ \beta = 2\tan(\alpha-\beta)$
ii) $\text{Cot}\ \beta = \tan\ \alpha - 2\tan\beta$.

If the plane is struck horizontally, show that tan α = 2tan β.

Solution:

The initial velocity and acceleration are split into components along the plane and $\perp^r$ to the plane as explained we have shown that the time of time of flight in $T = \dfrac{2u\text{Sin}\,(\alpha-\beta)}{g\,\text{Cos}\,\beta}$ (1)



Since the particle strikes the inclined plane normally its velocity parallel to the inclined normally, its velocity parallel to the inclined plane at the end of the time T is = O.

(ie) $O = u\ \text{Cos}\ (\alpha-\beta) - g\,\text{Sin}\,\beta \quad T$

Or $T = \dfrac{u\,\text{Cos}\,(\alpha-\beta)}{g\,\text{Sin}\,\beta}$ (2)

Equating (1) and (2)

We have

$$\frac{2u\,\text{Sin}\,(\alpha-\beta)}{g\,\text{Cos}\beta} = \frac{u\text{Cos}\,(\alpha-\beta)}{g\,\text{Sin}\,\beta}$$

(ie) $\text{Cot } \beta = 2\tan(\alpha - \beta)$ (i)

ie) $\cot \beta = \dfrac{2(\tan\alpha - \tan\beta)}{1+\tan\alpha\tan\beta}$

Cross multiplying

$\text{Cot } \beta + \tan\alpha = 2\tan\alpha - 2\tan\beta$ (or)

$\text{Cot } \beta = \tan\alpha - 2\tan\beta$

If the plane is stuck horizontally, the vertical velocity of the projectile at the end of time T is = O.

Initial vertical velocity = u sin α

And acceleration in this direction = g downwards

Vertical velocity in time T = u sin α - gT

$\therefore u\,\text{Sin}\,\alpha - gT = O \text{ or } T = \dfrac{u\,\text{Sin}\,\alpha}{g}$ (3)

Equating (1) and (3)

We have

$$\frac{2u\,\text{Sin}(\alpha-\beta)}{g\,\text{Cos}\,\beta} = \frac{u\,\text{Sin}\,\alpha}{g}$$

Or $2\,\text{Sin}(\alpha-\beta) = \text{Sin}\,\alpha\,\text{Cos}\,\beta$

(ie) $2(\sin\alpha\ \text{Cos}\,\beta - \text{Cos}\,\alpha\,\text{Sin}\,\beta) = \text{Sin}\,\alpha\,\text{Cos}\,\beta$

(ie) $\text{Sin}\,\alpha\ \cos\beta = 2\,\text{Cos}\,\alpha\,\text{Sin}\,\beta$ or $\tan\alpha = 2\tan\beta$.

Example: 4

If the greater range down an inclined plane is three time the greatest range up the plane, show that the plane is inclined at 30^0 to the horizon.

Solution:

Let u be the velocity of projection and β, the inclination of the plane to the horizontal.

Greatest range down the plane = $\dfrac{u^2}{g(1-\text{Sin}\,\beta)}$

Greatest range up the plane = $\dfrac{u^2}{g(1+\sin\beta)}$

Given that, greatest range down the plane

= 3 x greatest range up the plane.

Space for Hints

$$\Rightarrow \frac{u^2}{g(1-\sin\beta)} = 3.\ \frac{u^2}{g(1+\sin\beta)}$$

$$\Rightarrow 1+\sin\beta = 3(1-\sin\beta)$$

$$\Rightarrow 4\sin\beta = 2$$

$$\Rightarrow \sin\beta = \frac{1}{2}$$

$$\therefore \beta = 30^0$$

Example: 5

A particle projected with velocity u strikes at right angles a plane through the paint of projection inclined at an angle β to the horizon. Show that

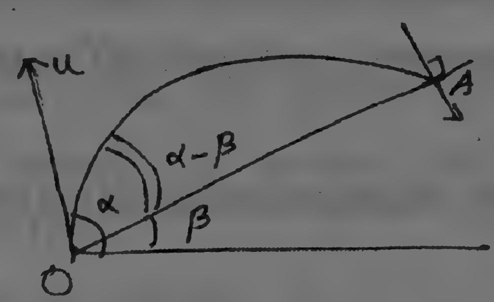
(i) The height of the point struck above the horizontal plane through the point of projection is $\dfrac{2u^2\sin^2\beta}{g\left[1+3\sin^2\beta\right]}$

(ii) Time of flight is $\dfrac{2u}{g\sqrt{[1+3\sin^2\beta]}}$.

(iii) The range of the particle on the inclined plane is $\dfrac{2u^2\sin\beta}{g[1+3\sin^2\beta]}$.

Proof:

Let O be the point of projection, α the angle of projection and A the point where it strikes the plane at right angle.



W.K.T Components of u along and perpendicular to the plane are u $\cos(\alpha-\beta)$ and u $\sin(\alpha-\beta)$ respectively. Let t be the time from o to A, then t is the time of flight.

$$\therefore\ t = \frac{2u\sin(\alpha-\beta)}{g\cos\beta} \qquad (1)$$

As the particle strikes the plane at A at right angle.

$\therefore$ Its velocity along the plane vanishes when it reaches A.

Considering motion along the inclined plane from o to A.

Initial velocity = u cos $(\alpha-\beta)$.

Final velocity = o, time = t.

Also acceleration up the plane = - g sin β.

Using "v = u + ft", we have

$$O = u\cos(\alpha-\beta) - g\sin\beta .\ t$$

$$\Rightarrow t = \frac{u\,\text{Cos}(\alpha-\beta)}{g\,\text{Sin}\,\beta} \qquad (2)$$

$\therefore$ From (1) and (2),

$$\frac{2u\,\text{Sin}(\alpha-\beta)}{g\,\text{Cos}\,\beta} = \frac{u\,\text{Cos}(\alpha-\beta)}{g\,\text{Sin}\,\beta}$$

$$\Rightarrow \frac{\text{Sin}(\alpha-\beta)}{\text{Cos}\,\beta} = \frac{\text{Cos}(\alpha-\beta)}{2\,\text{Sin}\,\beta}$$

$$= \sqrt{\frac{\sin^2(\alpha-\beta)+\text{Cos}^2(\alpha-\beta)}{\sqrt{\cos^2\beta+4\,\text{Sin}^2\beta}}}$$

$$\therefore \text{if } \frac{a}{b}=\frac{c}{d} \text{ then each} = \frac{\sqrt{a^2+c^2}}{\sqrt{b^2+d^2}}$$

$$= \frac{1}{\sqrt{1-\sin^2\beta+4\,\text{Sin}^2\beta}}$$

$$= \frac{1}{\sqrt{1+3\,\text{Sin}^2\beta}}$$

$$\therefore \frac{\text{Sin}(\alpha-\beta)}{\text{Cos}\,\beta} = \frac{\text{Cos}(\alpha-\beta)}{2\,\text{Sin}\,\beta} = \frac{1}{\sqrt{1+3\,\text{Sin}^2\beta}}$$

Taking Ist and IIIrd ratio

$$\therefore \frac{\text{Sin}(\alpha-\beta)}{\text{Cos}\,\beta} = \frac{1}{\sqrt{1+3\,\text{Sin}^2\beta}}$$

$$\Rightarrow \text{Sin}(\alpha-\beta) = \frac{\text{Cos}\,\beta}{\sqrt{1+3\,\text{Sin}^2\beta}} \qquad (3)$$

Taking 2nd and IIIrd ratio

$$\frac{\text{Cos}(\alpha-\beta)}{2\,\text{Sin}\,\beta} = \frac{1}{\sqrt{1+3\,\text{Sin}^2\beta}}$$

$$\Rightarrow \text{Cos } (\alpha-\beta) = \frac{2\,\text{Sin}\,\beta}{\sqrt{1+3\sin^2\beta}} \qquad (4)$$

(ii) Time of flight $= \dfrac{2u\,\text{Sin}\,(\alpha-\beta)}{g\cos\beta}$

$$= \frac{2u}{g\,\text{Cos}\,\beta} \text{ x } \frac{\text{Cos}\,\beta}{\sqrt{1+3\text{Sin}^2\beta}} (\therefore \text{by}(3))$$

$$= \frac{2u}{g\sqrt{1+3\sin^2\beta}}$$

(iii) Range on the plane $= \dfrac{2u^2\,\text{Sin}\,(\alpha-\beta)\,\text{Cos}\,\alpha}{g\,\text{Cos}^2\beta}$

$$= \frac{2u^2\,\text{Sin}\,(\alpha-\beta)}{g\,\text{Cos}^2\beta}.\ \text{Cos } (\alpha-\beta+\beta)$$

$$= \frac{2u^2\,\text{Sin}\,(\alpha-\beta)}{g\,\text{Cos}^2\beta}.\ [\text{Cos}\,(\alpha-\beta)\,\text{Cos}\,\beta - \sin\,(\alpha-\beta)\sin\beta]$$

$$= \frac{2u^2\sin(\alpha-\beta)}{g\,\text{Cos}^2\beta}. \qquad \frac{\text{Cos}\,\beta}{\sqrt{1+3\sin^2\beta}}$$

$$\text{x}\left[\frac{2\,\text{Sin}\,\beta}{\sqrt{1+3\text{Sin}^2\beta}}\cos\beta - \frac{\text{Cos}\,\beta}{\sqrt{1+3\sin^2\beta}}\sin\beta\right]$$

$$= \frac{2u^2\,\text{Sin}\,\beta}{g\sqrt{1+3\text{Sin}^2\beta}}\left[\frac{2-1}{\sqrt{1+3\text{Sin}^2\beta}}\right]$$

$$= \frac{2u^2}{g}.\frac{\sin\beta}{1+3\sin^2\beta}$$

(i) Let R be the range on the inclined plane.

Then $R = \dfrac{2u^2}{g}.\dfrac{\sin\beta}{1+3\text{Sin}^2\beta}$ $\qquad (\therefore \text{by(ii)})$

Vertical height of the point struct = R sin β

$$= \frac{2u^2}{g}.\frac{\text{Sin}^2\beta}{1+3\sin^2\beta}.$$

Example: 6

A particle is projected with speed u so as to strike at right angles a plane through the point of projection inclined at 30^0 to the horizon. Show that the range on this inclined plane is $\frac{4u^2}{7g}$.

Solution:

Let α be the angle of projection.

Let the particle strike the plane at right

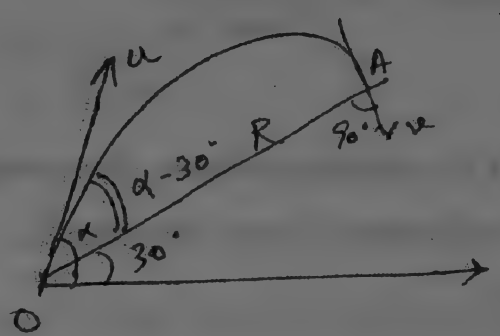
Angles at A.

Let OA = R and let t be the time of flight.

$$\text{Then } t = \frac{2u\,\text{Sin}(\alpha - 30^0)}{g\,\text{Cos}30^0}$$

$$t = \frac{4u\sin(\alpha - 30^0)}{v3g} \qquad (1)$$

$$(\therefore \text{Cos}30^0 = \frac{\sqrt{3}}{2})$$



For motion up the plane from O to A.

Initial velocity at O = 4 cos $(\alpha - 30^0)$

Final velocity at A = O

Acceleration up the plane = - g sin 30^0 = $-\frac{1}{2}$g.

$$\text{Distance OA} = R, \qquad \text{Time} = \frac{4u\sin(\alpha - 30^0)}{\sqrt{3}g}$$

$$(\therefore \text{by}(1))$$

Using "v = u + ft"

$$\Rightarrow O = 4\,\text{Cos}(\alpha - 30^0) - \frac{1}{2}g\frac{4u\,\text{Sin}(\alpha - 30^0)}{\sqrt{3}\,g}$$

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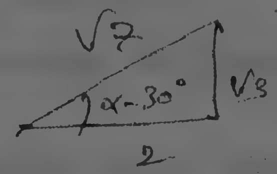
$$\Rightarrow \text{Cos}\left(\alpha-30^0\right)=\frac{2\,\text{Sin}\left(\alpha-30^0\right)}{\sqrt{3}}$$

$$\Rightarrow \tan\left(\alpha-30^0\right)=\frac{\sqrt{3}}{2}$$



Now using $v^2 - u^2 = 2fs$

$$\Rightarrow O-u^2\,\text{Cos}^2\left(\alpha-30^0\right)=-2.\frac{1}{2}gR$$

$$\Rightarrow R=\frac{u^2}{g}\text{Cos}^2\left(\alpha-30^0\right)$$

$$=\frac{u^2}{g}.\left(\frac{2}{\sqrt{7}}\right)^2$$

$$\therefore R=\frac{4u^2}{7g}.$$

Example: 7

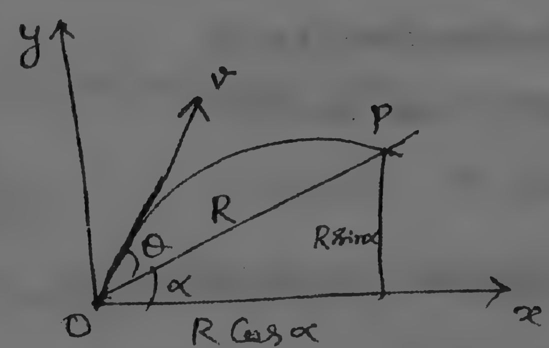
A shot is fired from a paint on an inclined α with a velocity v at an elevation $\alpha+\theta$, so as to move in a vertical plane through a line of greatest slope. If the shot hits the slope horizontally,

Show that $\tan\theta = \dfrac{\text{Sin}\,\alpha\,\text{Cos}\,\alpha}{1+\sin^2\alpha}$.

Proof:

If R is the range OP on the inclined plane such that at P the particle moves horizontally, with reference to axes through O,

We have $y = o$ at P.



$$Y = v\ \text{sim}\ (\theta+\alpha) - gt = O$$

$$\Rightarrow t=\frac{v\sin(\theta+\alpha)}{g} \qquad (1)$$

Also during the same time its displacement perpendicular to the plane is Zero.

W.K.T "$S = ut + \frac{1}{2} f t^2$"

$$S = v \sin \theta . \ t - \frac{1}{2} g \cos \alpha \, t^2$$

S = 0, for t = 0 which is the instant of projection

And for $t = \dfrac{2v \sin \theta}{g \cos \alpha}$ (2)

From (1) and (2) we set that

$$\frac{2v \sin \theta}{g \cos \alpha} = \frac{v \sin(\theta + \alpha)}{g}$$

$$\Rightarrow 2 \sin \theta = \cos \alpha \sin(\theta + \alpha)$$

$$\Rightarrow 2 \sin \theta = \cos \alpha [\sin \theta \cos \alpha + \sin \alpha \cos \theta]$$

$$\Rightarrow 2 \sin \theta = \sin \theta \cos^2 \alpha + \sin \alpha \cos \theta \cos \alpha$$

$$\Rightarrow 2 \sin \theta - \sin \theta \cos^2 \alpha = \sin \alpha \cos \alpha \cos \theta$$

$$\Rightarrow \sin \theta (2 - \cos^2 \alpha) = \sin \alpha \cos \alpha \cos \theta.$$

$$\Rightarrow \tan \theta = \frac{\sin \alpha \cos \alpha}{2 - \cos^2 \alpha}$$

$$\therefore \tan \theta = \frac{\sin \alpha \cos \alpha}{1 + \sin^2 \alpha}$$

$$[\because 2 - \cos^2 \alpha = 1 + 1 - \cos^2 \alpha = 1 + 1 \sin^2 \alpha]$$

Example: 8

Show that greatest range on an inclined plane through the point of projection is equal to the distance through which the particle could fall freely during the corresponding time of flight.

Proof:

Let u be the velocity of projection. Let the plane be inclined at an angle β to the horizon.

For maximum range up the plane, the angle of projection $\alpha = \frac{\pi}{4} + \frac{\beta}{2}$.

$$T = \text{time of flight up the plane} = \frac{2u \sin(\alpha - \beta)}{g \cos \beta}$$

$$= \frac{2u \operatorname{Sin}(\frac{\pi}{4}+\frac{\beta}{2}-\beta)}{g \operatorname{Cos}\beta} = \frac{2u \operatorname{Sin}(\frac{\pi}{4}-\frac{\beta}{2})}{g \operatorname{Cos}\beta}$$

Distance through which the particle would fall freely during time of flight up the plane $= \frac{1}{2} g T^2$

$$= \frac{1}{2} g . \frac{4u^2 \operatorname{Sin}^2(\frac{\pi}{4}-\frac{\beta}{2})}{g^2 \operatorname{Cos}^2\beta}$$

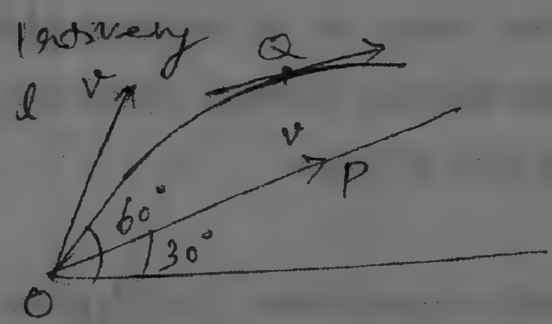
$$= \frac{u^2}{g} . \frac{2 \operatorname{Sin}^2(\frac{\pi}{4}-\frac{\beta}{2})}{\operatorname{Cos}^2\beta}$$

$$= \frac{u^2}{g} . \frac{1-\operatorname{Cos}(\frac{\pi}{2}-\beta)}{\operatorname{Cos}^2\beta} = \frac{u^2}{g} \frac{(1-\sin\beta)}{1-\sin^2\beta}$$

$$= \frac{u^2}{g(1+\operatorname{Sin}\beta)} = \text{maximum range up the plane.}$$

Example: 9

Two particles are projected simultaneously one with a velocity v up a smooth plane inclined at an angle of 30^0 to the horizon, and the other with a velocity $\frac{2v}{\sqrt{3}}$ at an elevation of 60^0. Show that the particles will be relatively at rest at the end of $\frac{2v}{3g}$ seconds



Proof:

The two particles will be relatively at rest if they move with equal velocities along parallel lines after a certain time.

Let P be the particle moving along the plane and Q, in a parabolic path.

Let their directions of motion be parallel after a time t.

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$\therefore$ After a time t, direction of Q makes an angle of 30^0 with the horizontal.

Using $\tan\theta = \dfrac{u \sin\alpha - gt}{u\cos\alpha}$, we have

$$\Rightarrow \tan 30^\circ = \frac{\frac{2v}{\sqrt{3}}\sin 60^\circ - gt}{\frac{2v}{\sqrt{3}}\cos 60^\circ}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \left(\frac{2v}{\sqrt{3}}\frac{\sqrt{3}}{2} - gt\right) \Big/ \frac{2v}{\sqrt{3}}.\frac{1}{2}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{v-gt}{v/\sqrt{3}}$$

$$\Rightarrow v - gt = \frac{v}{3}$$

$$\therefore t = \frac{2v}{3g}$$

Velocity of P after time $\dfrac{2v}{3g}$

$$= v - g\sin 30^0 . \frac{2v}{3g}$$

$$[\because v = u - g\sin\alpha.t]$$

$$= v - \frac{1}{2}.\frac{2v}{3} = \frac{2v}{3}$$

Velocity of Q after time $\dfrac{2v}{3g}$

$$= \sqrt{\frac{4u^2}{3} - 2\frac{2v}{\sqrt{3}}.\sin 60^0.g.\frac{2v}{3g} + g^2.\frac{4v^2}{9g^2}}$$

$$\left[\because v = \sqrt{u^2 - 2u\sin\alpha.gt + g^2t^2}\right]$$

$$= \sqrt{\frac{4u^2}{3} - \frac{8v^2}{3\sqrt{3}}.\frac{\sqrt{3}}{2} + \frac{4v^2}{9}}$$

$$= \sqrt{\frac{4u^2}{3} - \frac{4v^2}{3} + \frac{4u^2}{9}} = \frac{2v}{3}$$

$\therefore$ The particles are moving with equal velocities in parallel lines after time $\frac{2v}{3g}$.

Hence the particles are relatively at rest after time $\frac{2v}{3g}$.

Exercises:

1) A particle is projected form the top of a plane inclined at 60^0 to the horizontal. If the direction of projection is i) 30^0 above the horizontal and ii) 30^0 below the horizontal, show that the range down the plane in the first case is doubles that in the second.

2) The greatest range with a given velocity of projection an a horizontal plane is 3000 meters. Find the greatest ranges up and down a plane inclined at 30^0 to the horizon.

7.2 Impact:

Introduction:

A solid body has a definite shape. When a force in applied at any point of it tending to change its shape, in general, all solids which we meet with in nature yield slightly and get more or less deformed near the point. Immediately, internal forces come into play tending to restore the body to its original form and as soon as the disturbing force in removed, the body regains its original shape. The internal force which acts, when a body tends to recover its original shape after a deformation or compression is called the force of restitution. Also the property which causes a solid body to recover its shape is called elasticity. If a body does not tend to recover its shape, it will cause no force of restitution and such a body is said to be inelastic.

Suppose a ball is dropped from any height h upon a hard floor. It strikes the floor with a velocity $u = \sqrt{2gh}$ and makes an impact. Soon it rebounds and moves vertically upwards with a velocity v. The height h_1, to which it rebounds is given by $h_1 = \frac{v^2}{2g}$.(ie) $v=\sqrt{2gh_1}$.

Generally we find that $h_1 < h$. So $v < u$. As soon as the ball strikes, the floor, the impulsive action of the floor rapidly stops the downward velocity of the ball and at the same time causes a temporary compression near the point of contact. Due to the elastic property of the solid, the ball tends to regain its original from quickly. It presses the floor and receives an equal and opposite impulsive reaction from it and with a new upward velocity, it rebounds.

Now, bodies made of various materials are elastic in different degrees. If balls of different materials (like, iron, glass, lead, etc) are dropped from the same height upon a floor or if the same ball is dropped, upon floors of different constitution (like wooden floor, marble floor etc), It will be fond that the heights to which they rebound after striking the floor will be different. In all these cases, the velocity of the ball on reaching the floor is the same, as it is dropped from the same height. But the velocity of the ball after impact is not the same in each case as the height to which it rebounds is different. Thus due to the elastic property of solid bodies, a change in velocity takes place when strike each other.

If $v = u$, the velocity with which the ball leaves the floor is the same as that with which it strikes it. In this case, the ball is said to be perfectly elastic. If $v= 0$, the ball does not rebound at all. It is said to be inelastic. More generally, when a body completely regains is shape after a collision, it is said to be perfectly elastic. If it does not come to its original shape, it is said to be perfectly inelastic. These two cases of bodies are only ideal.

In this chapter, we shall study some simple cases of the impact of elastic bodies. We shall consider the cases of particles in collision with particles, or planes and of sphere in collision with planes or spheres. In all cases, we consider the impinging bodies to be smooth, so that the only mutual action they can have on each other will be along the common normal at the point where they touch.

7.2.1 Definitions:

Two bodies are said to impinge directly when the direction of motion of each before impact is along the common normal at the point where they touch.

They are said to be impinge obliquely, if the direction of motion of either body or both is not along common normal at the point where they touch.

The common normal at the point of contact is called the line of impact. Thus in the case of two spheres, the line of impact in the line joining their centres.

Space for Hints

7.2.2 Fundamental Laws of Impact:

The following three general principles hold good when two smooth moving bodies make an impact.

1. Newton's Experimental Law:

Newton studied the rebound of elastic bodies experimentally and the result of his experiments is embodies in the following law;-

When two bodies impinge directly, their relative velocity after, impact bears a constant ratio to their relative velocity before impact, and is in the opposite direction. If two bodies impinge obliquely, their relative velocity resolved along their common normal after impact bears a constant ratio to their relative velocity before impact, resolved in the same direction and is of opposite sign.

The constant ratio depends on the material of which the bodies are made and is independent of their masses. It is generally denoted by e, and is called the coefficient (or modulus) of elasticity (or restitution or resilience).

This law can be put symbolically as follows:

If u_1, u_2 are the components of the velocity of two impinging bodies along their common normal before impact and v_1, v_2 their components velocities along the same line after impact, all components being measured in the same direction and e is the coefficient of restitution.

Then $\dfrac{v_2 - v_1}{u_2 - u_1} = -e.$

The quantity e, which is a positive number, is never greater than unity. It lies between 0 and 1. Its value differs widely for different bodies; for two glass balls it is about 0.9; for ivory 0.8; while for lead it is 0.2. For two balls, one of lead and the other of iron, its value is about 0.13. Thus, when one or both the bodies are altered, e becomes different but so long as both the bodies remain the same, e is constant. Bodies for which $e = 0$ are said to be inelastic while for perfectly elastic bodies, $e = 1$. Probably, there are no bodies in nature coming strictly under either of these headings. Newton's law is purely empirical and is true only approximately, like many experimental laws.

Space for Hints

Motion of two smooth bodies $\perp^r$ to the line of Impact:

When two smooth bodies impinge, the only force between them at the time of impact in the mutual reaction which acts along the common normal. There is no force acting along the common tangent and hence there is no changes of velocity in that direction. Hence the velocity of either body resolved in a direction $\perp^r$ to the line of impact is not attached by impact.

7.2.3 Principle of conservation of momentum:

We can apply the law of conservation of momentum in the case of two impinging bodies. The algebraic sum of the momenta of the impinging bodies after impact in equal to the algebraic sum of their momenta before impact all momenta, being measured along the common normal.

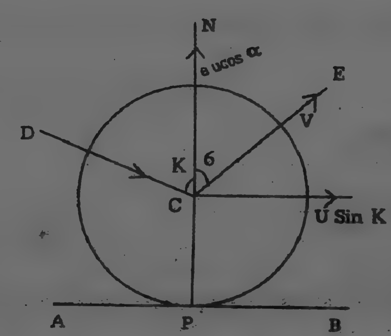
The above three principles are sufficient to study the changes in the motion of two impinging elastic bodies.

We shall now proceed to discuss particular cases.

7.3 Impact on a fixed plane

7.3.1 Impact of a smooth sphere on a fixed smooth plane:

A smooth sphere, or particle, whose mass is m and whose coefficient of restitution is e, impinges obliquely on a smooth fixed plane: to find its velocity and direction of motion after impact.



Let AB be the plane and P the point at which the sphere strikes it. The common normal at P is the vertical line at P passing through the centre of the sphere. Let it be PC. This is the line of impact. Let the velocity of the sphere before impact be u at an angle α with CP and v its velocity after impact at an angle θ with CN as shown in the figure. Since the plane and the sphere are smooth, the only force acting impact is the figure. Since the plane and the sphere are smooth, the only force acting impact is the implusive reaction and this is along the common normal. There is no force parallel to the plane during impact. Hence

Space for Hints

the velocity of the sphere resolved in a direction parallel to the plane is unaltered by the impact.

Hence $v \sin\theta = u \sin\alpha$ (1)

By Newton's experimental law, the relative velocity of the sphere along the common normal after impact is (-e) times its relative velocity along the common normal before impact. Hence

$v \cos\theta - 0 = -e(-u\cos\alpha - 0)$

(ie) $v\cos\theta = eu\cos\alpha$ (2)

Squaring (1) and (2) and adding

We have

$$v^2 = u^2(\mathrm{Sin}^2\alpha + e^2\cos^2\alpha)$$

(ie) $v = u\sqrt{\mathrm{Sin}^2\alpha + e^2\,\mathrm{Cos}^2\alpha}$ (3)

Dividing (2) by (1)

We have $\mathrm{Cot}\,\theta = e\,\mathrm{Cot}\,\alpha$ (4)

Give the velocity and direction of motion after impact

Corrollary: 1

If $e = 1$, we find that from (3), $v = u$ and from (4), $\theta = \alpha$,

Hence if a perfectly elastic sphere impinges on a fixed smooth plane, its velocity is not altered by impact and the angle of reflection is equal to the angle of incidence.

Corrollary: 2

If $e = 0$, then from (2)

$v\cos\theta = 0$ and

From (3), $v = u\sin\alpha$

Hence Cos = 0.

(ie) $\theta = 90^0$

Hence the inelastic sphere slides along the plane with velocity $u\sin\alpha$.

Corrollary: 3

If the impact is direct, we have $\alpha = 0$. Then $\theta = 0$ and from (3),

$v = eu$

Hence if an elastic sphere strikes a plane normally with velocity u, it will rebound in the same direction with velocity eu.

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Corollary: 4

The impulse of the pressure on the plane is equal and opposite to the impulse of the pressure on the sphere. The impulse I on the sphere is measured by the change in momentum of the sphere along the common normal.

$$1 = mv \text{ Cos } \theta - (-mu \text{ Cos } \alpha) = m(r \cos \theta + u \cos \alpha)$$

$$= m\,(eu \text{ Cos } \alpha + u \text{ Cos } \alpha) = mu \text{ Cos } \alpha(1 + e)$$

Corollary: 5

Loss of kinetic energy due to the impact

$$= \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$

$$= \frac{1}{2}mu^2 - \frac{1}{2}mu^2\,(\text{Sin}^2\alpha + e^2 \text{ Cos}^2\alpha)$$

$$= \frac{1}{2}mu^2\,(1 - \text{Sin}^2\alpha - e^2 \text{ Cos}^2\alpha) = \frac{1}{2}mu^2(\cos^2\alpha - e^2\cos^2\alpha)$$

$$= \frac{1}{2}(1 - e^2)\,mu^2 \text{ Cos}^2\alpha$$

If the sphere is perfectly elastic.

e = 1 and loss of kinetic energy is zero.

Example: 1

A Smooth circular table is surrounded by a smooth rim whose interior surface is vertical. Show that a ball projected along the table from a point A on the rim in a direction making an angle α with the radius through A will return to the point of projection after two impacts if

$$\tan \alpha = e^{(3/2)} \Big/ \sqrt{1+e+e^2}$$

Prove also that, when the ball returns to the point of projection, its velocity is to its original velocity as $e^{\left(\frac{3}{2}\right)} : 1$

Solution:

Let the ball starting from A return to it after two reflections at B and C. At B, the point of the first impact, the common normal is the radius OB and at C, the point of the second impact, the common normal is OC.

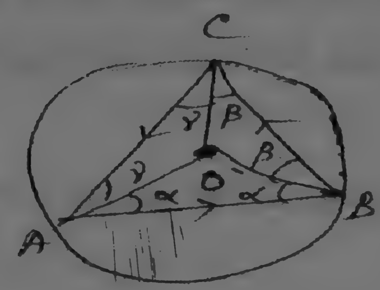
Let $\angle OBC = \beta$ and $\angle OCA = \gamma$.

Then $\angle OCB = \beta$ and $\angle OAC = \gamma$

Considering the impact at B, and applying equation (4). Of section 7.3.1 impact of a smooth sphere.

We have $\text{Cot } \beta = e \text{Cot}\, \alpha$

(ie) $\tan \beta = \frac{1}{e} \tan \alpha$ (1)



Similarly,

Considering the impact at C,

$\text{Cot } \gamma = e \text{ Cot } \beta.$

$\therefore \tan \gamma = \frac{1}{e} \tan \beta = \frac{1}{e^2} \tan \alpha$ (2)

Now, In $\Delta ABC, \angle A + \angle B + \angle C = 2\alpha + 2\beta + 2\gamma = 180^0$

(ie) $\therefore \alpha + \beta + \gamma = 90^0$ or $\alpha = 90^0 - (\beta + \gamma)$

$\therefore \quad \alpha = \tan(90^0 - \overline{\beta+\gamma}) = \text{Cot}(\beta+\gamma)$

$$= \frac{1}{\tan(\beta+\gamma)} = \frac{1 - \tan\beta \tan\gamma}{\tan\beta + \tan\gamma}$$

(ie) $\tan \alpha(\tan\beta + \tan\gamma) = 1 - \tan\beta \tan\gamma.$

(ie) $\tan \alpha\left(\frac{1}{e} \tan\alpha + \frac{1}{e^2} \tan\alpha\right) = 1 - \frac{1}{e} \tan\alpha \,.\, \frac{1}{e^2} \tan\alpha$

(ie) $\tan \alpha\left(\frac{1}{e} + \frac{1}{e^2}\right) = 1 - \frac{\tan^2 \alpha}{e^3}$ [$\because$ by (1) & (2)]

(ie) $\tan^2 \alpha\left(\frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3}\right) = 1$

(or) $\tan^2 \alpha\left(\frac{1+e+e^2}{e^3}\right) = 1$

(ie) $\tan^2 \alpha = \frac{e^3}{1+e+e^2}$ (3)

(or)

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$$\tan\alpha = \frac{e^{(3/2)}}{\sqrt{1+e+e^2}}$$

Let u be the velocity of projection from A,v be the velocity of the ball after the first impact at B. and w be the velocity after the second impact at C.

Applying equation (3)

$v^2 = u^2 (\text{Sin}^2\alpha + e^2 \text{Cos}^2\alpha)$ and

$w^2 = v^2 (\text{Sin}^2\beta + e^2 \text{Cos}^2\beta)$

$w^2 = u^2 (\text{Sin}^2\alpha + e^2 \text{Cos}^2\alpha)\ (\text{Sin}^2\beta + e^2 \text{Cos}^2\beta)$

$= u^2 \text{Cos}^2\alpha (\tan^2\alpha + e^2)\ \text{Cos}^2\beta\ (\tan^2\beta + e^2)$

$$= \frac{u^2 (\tan^2\alpha + e^2)(\tan^2\beta + e^2)}{(1+\tan^2\alpha)(1+\tan^2\beta)}$$

$$= \frac{u^2(\tan^2\alpha + e^2)(\frac{1}{e^2}\tan^2\alpha + e^2)}{(1+\tan^2\alpha)(1+\frac{1}{e^2}\tan^2\alpha)} \text{ using } (1)$$

$$= \frac{u^2 (\tan^2\alpha + e^2)(\tan^2\alpha + e^4)}{(1+\tan^2\alpha)(e^2 + \tan^2\alpha)}$$

$$= \frac{u^2(\tan^2\alpha + e^4)}{(1+\tan^2\alpha)}$$

$$= \frac{u^2\left(\frac{e^4}{1+e+e^2} + e^4\right)}{\left[1+\frac{e^4}{1+e+e^2}\right]} \text{ Susbstituting from (2)}$$

$$\because w^2 = \frac{u^2 (e^3+e^4+e^5+e^6)}{1+e+e^2+e^3} = u^2 e^3$$

$w = u.e^{(3/2)}$ or $w : u = e^{(3/2)} : 1$

Example: 2

A particle falls from height h upon a fixed horizontal plane: it e be the coefficient of the restitution. Show that the whole distance described before the particle has finished rebounding is $h\left(\frac{1+e^2}{1-e^2}\right)$ Show also that the whole time taken is $\frac{1+e}{1-e}\sqrt{\frac{2h}{g}}$.

Space for Hints

Solution:

Let the velocity of the particle on first hitting the plane. Then $u^2 = 2gh$.

After the first impact, the particle rebounds with a velocity eu and ascends a certain height, retraces its path and makes a second impact with the plane with velocity eu. After the second impact, it rebounds with a velocity e^2 u and the process is repeated a number of times. The velocities after the thirds, fourth etc.

Impacts are e^3u, e^4 u etc. The height ascended after the first impact with velocity $eu = \frac{(\text{velocity})^2}{2g} = \frac{e^2u^2}{2g}$.

The height ascended after the second impact with velocity $e^2u = e^4 u^2/2g$ and so on.

$\therefore$ Total distance travelled before the particle stops rebounding

$$= h + 2\left(\frac{e^2u^2}{2g} + \frac{e^4u^2}{2g} + \frac{e^6u^2}{2g} + \ldots\ldots\right)$$

$$= h + \frac{2.e^2u^2}{2g}(1 + e^2 + e^4 + \ldots\ldots\ldots\ldots \text{to} \infty)$$

$$= h + \frac{e^2u^2}{g}.\frac{1}{1-e^2} = h + \frac{e^2.2gh}{h}.\frac{1}{1-e^2}$$

$$= h\left(1 + \frac{2e^2}{1-e^2}\right) = h\left(\frac{1+e^2}{1-e^2}\right)$$

Considering the motion before the first impact we have the initial velocity = 0, acceleration = g, final velocity = u and so if t is the time taken,

$u = O + gt$.

$$\therefore\ t = \frac{u}{g} = \frac{\text{velocity}}{g}.$$

Time interval between the first and second impact is

= 2 x time taken for gravity to reduce the velocity eu to O.

= 2. velocity /g = 2eu/g.

Similarly time interval between the second and third and impacts = $2e^2$ u/g and so on.

So total time taken

$$=\frac{u}{g}+2\left(\frac{eu}{g}+\frac{e^2u}{g}+\frac{e^3u}{g}+\text{..........to}\infty\right)$$

$$=\frac{u}{g}+\frac{2eu}{g}(1+e+e^2+\text{............to}\infty)=\frac{u}{g}+\frac{2eu}{g}.\frac{1}{1-e}=\frac{u}{g}\left[1+\frac{2e}{1-e}\right]$$

$$\frac{u}{g}\left(\frac{1+e}{1-e}\right)=\frac{\sqrt{2gh}}{g}\left(\frac{1+e}{1-e}\right)$$

Total time = $\left(\frac{1+e}{1-e}\right)\sqrt{\frac{2h}{g}}.$

Example: 3

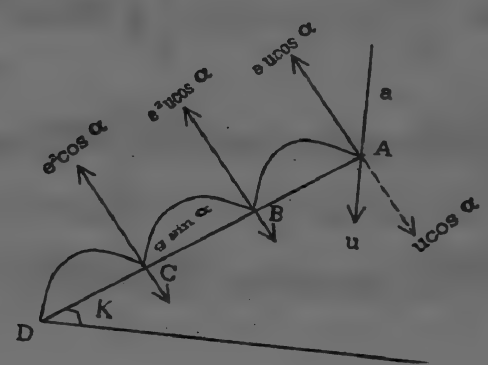
A particle of elasticity 'e' is dropped from a vertical height 'a' upon the highest point of a plane which is of length b and is inclined at an angle α to the horizon and descends to the bottom in three jumps. Show that

$b = 4ae\ (1 + e)\ (1 + e^2)\ (1 + e + e^2) \sin \alpha$

Solution:

The downwards vertical velocity at A before string = $\sqrt{2ga}$ and'et this=u.

This can be resolved into



Two components as u cos $\alpha \perp^r$ to the inclined plane and u sin α Parallel to the plane. At the impact at A, there is no force parallel to the plane and there is only the impulsive reaction normal to the plane. So the component u sin α is not affected by impact while the component u cos α is reversed as eu cos α. Hence the particle describes a parabola and strikes the plane at B with a velocity eu Cos $\alpha \perp^r$ to it after impact this is reversed as e^2u cos α. The particle strikes the eu cos $\alpha \perp^r$ to it after impact this is reversed as e^2 u cos α. The particle strikes the inclined plane at C with a velocity e^2u cos α normal to it, which is reversed as e^3u cos α.

Let t_1, t_2, t_3 be the time taken to describe the paths AB, BC and CD respectively.

Space for Hints

Consider the motion, $\perp^r$ to the inclined plane. Distance traveled is that direction in the time $t_1 = 0$. Applying the formula "$S = ut + ½ ft^2$"

We have

$$O = eu \cos\alpha . \ t_1 - ½ g \cos\alpha . \ t_1^2$$

$$t_1 = \frac{2eu \cos\alpha}{g \cos\alpha} = \frac{2eu}{g}.$$

Similarly, $t_2 = \dfrac{2e^2u}{g}$ and $t_3 = \dfrac{2e^3u}{g}$.

Hence the total time taken from A to D

$$= t_1 + t_2 + t_3 = \frac{2eu}{g}(1+e+e^2)$$

In this period, the particle has described a distance b down the inclined plane, starting with an initial velocity u sin α and acted on by an acceleration g sin α.

$$\therefore \ b = u \sin\alpha(t_1+t_2+t_3)+\frac{1}{2}g \sin\alpha(t_1+t_2+t_3)^2$$

$$\text{(ie) } b = u \sin\alpha . \ \frac{2eu}{g}(1+e+e^2)+\frac{1}{2}g \sin\alpha . \frac{4e^2u^2}{g^2} x (1+e+e^2)^2$$

$$= \frac{2eu \sin\alpha}{g}(1+e+e^2)+\frac{2e^2u^2 \sin\alpha}{g}(1+e+e^2)^2$$

$$= \frac{2eu \sin\alpha}{g}(1+e+e^2)+\frac{2e^2u^2 \sin\alpha}{g}(1+e+e^2)^2$$

$$= \frac{2eu^2 \sin\alpha}{g}(1+e+e^2)[1+e(1+e+e^2]$$

$$= \frac{2eu^2 \sin\alpha}{g}(1+e+e^2)(1+e+e^2+e^3)$$

$$= \frac{2e.2ga.\sin\alpha}{g}(1+e+e^2)(1+e)(1+e^2)$$

$$b = 4ae \sin\alpha(1+e)(1+e^2)(1+e+e^2)$$

Example: 4

A particle is projected from a point on an inclined plane and at the r^{th} impact, it strikes the plane $\perp^r$ and at the n^{th} impact is at the point of projection. Show that $e^n - 2e^r + 1 = 0$.

Solution:

Let α be the inclination of the plane to the horizontal and u the velocity of projection at an angle θ to the inclined plane. This velocity can be resolved into two components, u cos θ along the upward inclined plane and u sin $\theta \perp^r$ to the inclined plane. The acceleration g can be resolved into two components, g sin α along the downward inclined plane and g cos $\alpha \perp^r$ to the inclined plane and downwards.

Consider motion $\perp^r$ to the plane.

Let t_1 be the time upto the first impact Distance traveled $\perp^r$ to the plane in time t_1 is O. (ie) $O = u \sin \theta . \ t_1 - \frac{1}{2} g \cos \alpha . t_1^2$

$$\therefore t_1 = \frac{2u \sin \theta}{g \cos \alpha}.$$

The particle strikes the plane the first time with a velocity u sin $\theta \perp^r$ to it and after this impact this component is reversed as eu $\sin\theta$. Hence time interval between the first and second impacts.

$= 2eu \sin \theta / g \cos \alpha$.

The particle strikes the plane a second time with a velocity eu $\sin\theta \perp^r$ to it and after the second impact, this component is reversed as e^2 u Sin θ.

Hence time interval between the second and third impacts = $2e^2$ u sin θ/g Cos α and so on.

Time till the r^{th} impact

$$= \frac{2u \sin\theta}{g \cos\alpha} + \frac{2eu \sin\theta}{g \cos\alpha} + \frac{2e^2 u \sin\theta}{g \cos\theta} + \text{............to r terms.}$$

$$= \frac{2u \sin\theta}{g \cos\alpha} (1 + e + e^2 + \text{...........to r terms})$$

$$= \frac{2u \sin\theta}{g \cos\alpha} . \left(\frac{1-e^r}{1-e} \right) \qquad (1)$$

At the end of this time, the particle strikes the plane$\perp^r$. So the velocity parallel to the plane at that instant = 0

$$\text{Hence } u \cos\theta - g \sin \alpha . \frac{2u \sin\theta}{g \cos\alpha} . \frac{1-e^r}{1-e} = 0$$

Space for Hints

(ie) $\text{Cos }\theta \;\text{Cos }\alpha(1-e) = 2\,\text{Sin}\,\alpha\;\text{Sin }\theta\,(1 - e^r)$ (2)

Putting r = n is (1)

$$\text{Time till the } n^{th} \text{ impact} = \frac{2u\,\text{Sin}\,\theta}{g\,\text{Cos}\,\alpha}.\frac{1-e^n}{1-e}$$

Now, it is at the point of projection.

Hence the distance traveled parallel to the plane upto this time = 0.

$$\text{(ie) } u\cos\theta\,.\,\frac{2u\,\text{Sin}\,\theta}{g\,\text{Cos}\,\alpha}.\frac{1-e^n}{1-e}-\frac{1}{2}g\,\text{Sin}\,\alpha\left(\frac{2u\,\text{Sin}\,\theta}{g\,\text{Cos}\,\alpha}.\frac{1-e^n}{1-e}\right)^2=0.$$

$$\Rightarrow u\cos\theta-\frac{1}{2}g\sin\alpha.\frac{2u\sin\theta}{g\cos\alpha}.\frac{1-e^n}{1-e}=0$$

(ie) $\text{Cos }\theta\,.\,\text{Cos }\alpha(1-e) = \text{Sin }\alpha\;\text{Sin }\theta\,(1 - e^n)$ (3)

Dividing (2) by (3)

We have

$$1=\frac{2(1-e^r)}{1-e^n}$$

(ie) $2 - 2e^r = 1 - e^n$

Or $e^n - 2e^r + 1 = 0$

Example: 5

A ball is thrown from a point on a smooth horizontal ground with a speed V at an angle α to the horizon. If e be the coefficient of restitution. Show that the total time for which the ball rebounds on the ground is $\frac{2V\text{Sin}\,\alpha}{g(1-e)}$ and the horizontal distance travelled by it is $\frac{V^2\sin 2\alpha}{g(1-e)}$

Solution:

The initial horizontal and vertical components of the velocity are $V\text{Cos}\,\alpha$ and $V\sin\alpha$. The particle describes a parabola and strikes the horizontal plane normally with velocity $u\sin\alpha$..

Due to impact, the horizontal velocity is not affected while the vertical component is reversed as $eV\,\text{Sin}\,\alpha$..

Similarly the vertical components of the velocity after the second, third etc., impacts are $e^2V\,\text{Sin}\,\alpha$ etc.

Let t_1, t_2, t_3 etc be the time for the successive trajectories.

$t_1 = \dfrac{2\ V\ \ Sin\ \ \alpha}{g}$, $t_2 = \dfrac{2eV\,Sin\,\alpha}{g}$, $t_3 = \dfrac{2e^2V\,Sin\,\alpha}{g}$ and so on. So total time that elapses before the particle stops rebounding.

$$= \frac{2VSin\,\alpha}{g} + \frac{2eV\,Sin\,\alpha}{g} + \frac{2e^2\,Sin\,\alpha}{g} + \ldots\ldots\ldots$$

$$= \frac{2VSin\,\theta}{g}\,(1+e+e^2+\ldots\ldots\ldots\text{to}\,\infty)$$

$$= \frac{2V\,Sin\,\theta}{g}\;\frac{1}{1-e} = \frac{2V\,Sin\,\alpha}{g(1-e)}$$

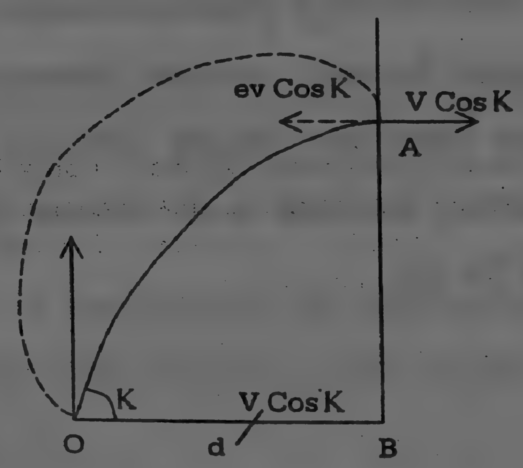
Throughout this time, the horizontal component V cos α is not affected. So horizontal distance described during this time $= V\cos\alpha . \dfrac{2V\sin\alpha}{g(1-e)} = \dfrac{V^2\sin 2\alpha}{g(1-e)}$

Example: 6

An elastic sphere is projected from a given point O with given velocity V at an inclination α to the horizontal and after hitting a smooth vertical wall at a distance d from O returns to O. Prove that $d = \dfrac{V^2\,Sin2\alpha}{g}.\dfrac{e}{1+e}$ Where e is the coefficient of restitution.

Solution:

Let the particle strike the wall at A. From O to A, the particle describes a parabola under gravity with constant horizontal velocity V cos α. Let t_1 be the time for this $V\cos\alpha\, t_1 = d$ (1)



At the impact at A there is no force parallel to the wall. The component V cos α being $\perp^r$ to the wall is reveised as eV Cos α. The particle will describe another parabola with constant horizontal velocity eV Cos α and return to O. Let t_2 be the time for this return journey. Then eV Cos $\alpha . T_2 = d$ (2).

Space for Hints

But the vertical motion is not affected by impact and throughout the interval $t_1 + t_2$ it is subject to retardation by g only.

As the particle returns to O, Vertical distance described in time $t_1 + t_2 = 0$

$$\therefore O = V \sin \alpha (t_1 + t_2) - \frac{1}{2} g (t_1 + t_2)^2 \text{ or}$$

$$\text{(or) } t_1 + t_2 = \frac{2V \sin \alpha}{g} \qquad (3)$$

Substituting for t_1, t_2 from (1) and (2) is (3).

We have

$$-\frac{d}{V \cos \alpha} + \frac{d}{eV \cos \alpha} = \frac{2V \sin \alpha}{g}$$

$$\text{(ie) } \frac{d(e+1)}{eV \cos \alpha} = \frac{2V \sin \alpha}{g}$$

$$\text{Or } d = \frac{2eV^2 \sin \alpha \cos \alpha}{g(1+e)}$$

$$\therefore d = \frac{V^2 \sin 2\alpha}{g} \cdot \frac{e}{1+e}$$

Exercises:

1) a) An elastic ball of mass m falls from a height h on a fixed horizontal plane and rebounds show that the loss of K.E. by the impact is mgh $(1 - e^2)$.

b) A particle falls from a height h is time t upon a fixed horizontal plane. Prove that it rebounds and reaches a maximum height e^2 h in time etc.

2) A particle dropped through a vertical distance of 20 meters, strikes the highest point of a plane which is of length 12metres and is inclined at 30^0 to the horizon. If the particle descends to the bottom is three jumps, Show that $e(1+e)(1+e^2)(1+e+e^{2)}) = 0.3$.

Space for Hints

UNIT – 8

DIRECT AND OBLIQUE IMPACT

8.1 Direct impact of two smooth spheres:

Definition:

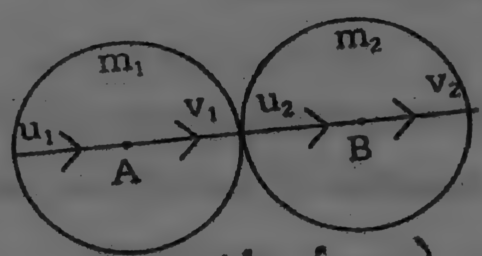
Direct impact

Two bodies are said to impinge directly if the direction of motion of each is along the common normal at the point of contact.

Result (1): A smooth sphere of mass m_1 impinges directly with velocity u_1 an another smooth sphere of mass m_2 Moving in the same direction with velocity u_2; if the coefficient of restitution is e, to find their velocities after the impact.

Proof:

AB is the line of impact, ie. the common normal. Due to the impact, there is no tangential force and hence, for either sphere the velocity along tangement is not altered by impact. But before impact, the spheres had been moving only along the line AB (as this is a case of direct impact). Hence for either sphere, tangential velocity after impact = its tangent velocity before impact = O. So after impact, the spheres will move only in the direction AB. Let their velocities be v_1 and v_2



fig·(8·1)

By newton's experimental law, the relative velocity of m_2 with respect to m_1 after impact is (-e) times the corresponding relative velocity before impact.

$$v_2 - v_1 = -e\,(u_2 - u_1) \qquad (1)$$

By the principle of conservation of momentum, the total momentum along the common normal after impact is equal to the total momentum in the same direction before impact.

$$\therefore m_1 v_1 + m_2 v_2 = m_1u_1 + m_2 u_2 \qquad (2)$$

(1) – (1) x m_2 gives

$v_1 (m_1 + m_2) = m_1v_1 + m_2u_2 + em_2 (u_2 - u_1)$

$= m_2u_2 (1 + e) + (m_1 - em_2)\, u_1.$

Space for Hints

$$\therefore v_1 = \frac{m_2u_2(1+e)+(m_1-em_2)u_1}{m_1+m_2} \qquad (3)$$

(1) $\times m_1$ + (2) gives

$v_2(m_1+m_2) = -em_1(u_2-u_1) + m_1u_1 + m_2u_2$

$= m_1u_1(1+e) + (m_2-em_1)u_2$

$$v_2 = \frac{m_1u_1(1+e)+(m_2-em_1)u_2}{m_1+m_2} \qquad (4)$$

Equation (3) and (4) give the velocities of the spheres after impact.

Note:

If one sphere, say m_2, is moving originally in a direction opposite to that of m_1 the sign of u_2 will be negative. Also it is most important that the directions of v_1 and v_2 must be specified clearly. Usually we take the positive direction as from left to right and then assume that both v_1 and v_2 are in the direction. If either of them is actually in the opposite direction, the value obtained for it will turn to the negative.

In writing equation (1) corresponding to Newton's law the velocities must be subtracted in the same order on the both sides. In all problems it is better to draw a diagram showing clearly the positive direction and the directions of the velocities of the bodies.

Corollary: 1

If the two spheres are perfectly elastic and of equal mass, then $e = 1$ and $m_1 = m_2$. Then from equations (3) and (4), we have

$$v_1 = \frac{m_1u_2 2+0}{2m_1} = u_2 \text{ and } v_2 = \frac{m_1u_1.2+0}{2m_1} = u_1.$$

i.e. if the two equal perfectly elastic spheres impinge directly, they interchange their velocities.

Corollary: 2

The impulse of the blow on the sphere

A of mass m_1 = Change of momentum of A

$= m_1(u_1 - u_2)$.

$$= m_1\left[\frac{m_2u_2(1+e)+(m_1-em_2)u_1}{m_1+m_2} - u_1\right]$$

$$= m_1\left[\frac{m_2u_2(1+e)+m_1u_1-em_2u_1-m_1u_1-m_2u_1}{m_1+m_2}\right]$$

$$= \frac{m_1[m_2u_2(1+e)-m_2u_1(1+e)]}{m_1+m_2} = \frac{m_1m_2(1+e)(u_2-u_1)}{m_1+m_2}$$

The impulsive blow on m_2 will be equal and opposite to the impulsive blow an m_2.

8.1.1: Loss of kinetic energy due to direct impact of two smooth spheres:

Two spheres of given masses with given velocities impinges directly; to show that there is a loss of kinetic energy and to find the amount.

Let m_1, m_2 be the masses of the spheres, u_1 and u_2 and v_1 and v_2 be their velocities before and after impact and e the coefficient of restitution.

By Newton's law,

$$v_2 - v_1 = -e(u_2 - u_1) \qquad (1)$$

By the principle of conservation of momentum.

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2 \qquad (2)$$

Total kinetic energy before impact $= \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2$

And total kinetic energy after impact $= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$

Change in K.E = initial K.E- Final K.E.

$$= \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 - \frac{1}{2}m_1v_1^2 - \frac{1}{2}m_2v_2^2$$

$$= \frac{1}{2}m_1(u_1-v_1)(u_1+v_1) - \frac{1}{2}m_2(u_2-v_2)(u_2+v_2)$$

$$= \frac{1}{2}m_1(u_1-v_1)(u_1+v_1) + \frac{1}{2}m_1(v_1-u_1)(u_2+v_2)$$

$$[\because m_2(u_2-v_2) = m_1(u_1-v_1) \text{ from } (2)]$$

$$= \frac{1}{2}m_1(u_1-v_1)[u_1+v_1-(u_2+v_2)]$$

$$= \frac{1}{2}m_1(u_1-v_1)[u_1-v_2-(u_2-v_1)]$$

$$= \frac{1}{2}m_1(u_1-v_1)[u_1-u_2+e(u_2-u_1)] \text{u sin g } (1)$$

$$=\frac{1}{2}m_1(u_1-v_1)(u_1-u_2)(1-e) \qquad (3)$$

Now from (2), $m_1(u_1-v_1)=m_2(v_2-u_2)$

$$\therefore \frac{u_1-v_1}{m_2}=\frac{v_2-u_2}{m_1} \text{ and each } =\frac{u_1-v_2+v_2-u_2}{m_1+m_2}$$

$$\text{ie. Each } = \frac{(u_1-u_2)+(v_2-v_1)}{m_1+m_2}$$

$$=\frac{(u_1-u_2)-e(u_2-u_1)}{m_1+m_2} \text{ usin (1)}$$

$$=\frac{(u_1-u_2)-(1+e)}{m_1+m_2}$$

$$\therefore u_1-v_1=\frac{m_2(u_1-u_2)(1+e)}{m_1+m_2} \text{ and substituting this in (3),}$$

$$\text{Change in K.E} = \frac{1}{2}\frac{m_1m_2(u_1-u_2)(1+e)(u_1-u_2)(1-e)}{m_1+m_2}$$

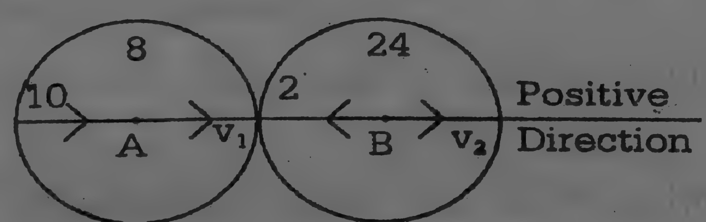
$$=\frac{1}{2}\frac{m_1m_2(u_1-u_2)^2(1-e^2)}{m_1+m_2} \qquad (4)$$

As e<1, the expression (4) is always positive and so the initial K.E. of the system is greater than the final K.E. So there is actually a loss of total K.E. by a collision. Only in the case, when e =1 (ie). Only when the bodies are perfectly elastic, the expression (4) becomes zero and hence the total K.E. is unchanged by impact.

Example: 1

A ball of mass 8gm. Moving with a velocity of 10cm per sec. Impinges directly on another of mass 24gm, moving at 2 cm per sec in the same direction. If e = ½ find the velocities after impact. Also calculate the loss in kinetic energy.

Solution:



Space for Hints

Let v_1 and v_2 cm per.sec. Be the velocities of the masses 8gm and 24gm respectively, after impact.

By Newton's law, $v_2 - v_1 = -\frac{1}{2}(2-10) = 4$ (1)

By the principle of momentum,

$24v_2 + 8\,v_1 = 24 \times 2 + 8 \times 10 = 128$

1. e $3\,v_2 + v_1 = 16$ (2)

Solving (1) and (2), v_1 = 1cm/sec., v_2 = 5cm/sec.

The K.E. before impact $= \frac{1}{2}.8.10^2 + \frac{1}{2}.24.2^2$

$= 448$ dynes

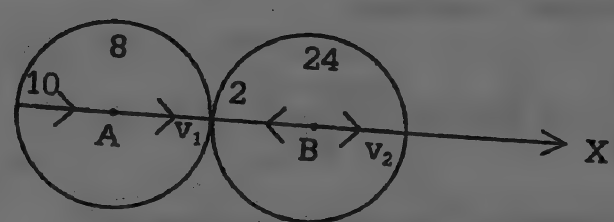
The K.E after impact $= \frac{1}{2}.8.1^2 + \frac{1}{2}.25.5^2 = 304$ dynes

$\therefore$ Loss in K.E = 144 dynes

Example: 2

If the 24gm,. mass in the previous question be moving in a direction opposite to that of the 8gm mass find the velocities after impact.

Solution:



Let v_1 and v_2 cm/sec. be the velocities of the 8gm and 24gms mass respectively after impact.

By Newton's law,

$v_2 - v_1 = -\frac{1}{2}(-2-10) = 6$ (1)

By conservation of momentum,

$24v_2 + 8v_1 = 24 \times (-2) + 8 \times 10 = 32$ ie.$3v_2 + v_1 = 4$ (2)

Solving (1) and (2), $v_1 = \frac{1}{2}$ cm / sec, $v_2 = \frac{5}{2}$ cm / sec.

The negative sign of v_1 shows that the direction of motion of the 8gm.mass is reversed as we had taken the direction left to right as positive and assumed v_1 to be in this direction. Since v_2 is positive the 24gm. ball moves from left to right after impact, so that its direction of motion is also reversed.

Example: 3

A ball overtakes another ball of m times its mass, which is moving with $\frac{1}{n}^{th}$ of its velocity in the same direction if the impact reduces the first ball to rest, prove that the coefficient of elasticity is $\frac{m+n}{m(n-1)}$.

Deduce that $m > \frac{n}{n-2}$.

Solution:

Taking AB in fig. (a). (*Ex* .1) as the positive direction. Let the mass of the first ball be k and u its velocity along AB before impact. Then, for the second ball, mass is mk and $\frac{u}{n}$ is the velocity before impact. After impact the first ball is reduced to rest and let v be the velocity of the second ball.

By Newton's law of impact, we have

$$v - O = -e\left(\frac{u}{n} - u\right) \text{(ie) } v = \frac{eu(n-1)}{n} \qquad (1)$$

By principle of conservation of momentum along AB,

$$K \times O + mk.v. = ku + mk.\frac{1}{n}u$$

$$\text{i.e } mv = u + \frac{m}{n}u = \frac{u(m+n)}{n} \qquad (2)$$

Substituting the value of v from (1) and (2), we have

$$\frac{meu(n-1))}{n} = \frac{u(m+n)}{n} \text{ or } e = \frac{(m+n)}{m(n-1)}$$

Now e is positive and less than 1,

$$\therefore m(n-1) > m+n \text{ i.e. } mn - 2m > n$$

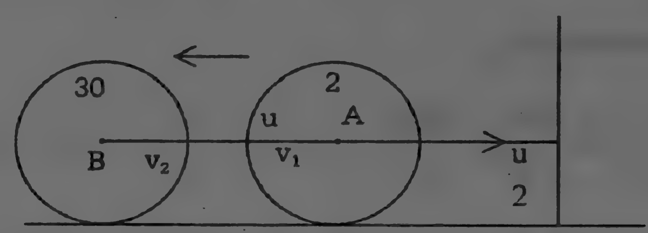
$$\therefore m(n-2) > n \text{ or } m > \frac{n}{n-2}$$

Space for Hints

Example: 4

Two equal sphere A and B of masses 2gm, and 30gm respectively lie on a smooth floor, So that their line of centers is perpendicular to a fixed vertical wall. A being nearer to the wall. A is projected to words B. Show that, if the coefficient of restitution between the two spheres and that between the first sphere and the wall is $\frac{3}{5}$, then A will be reduced to rest after its second impact with B.

Solution:



Consider the impact between A and B. Taking AB as the positive direction,

Let the velocity of A before impact be u. B is at rest.

After the impact, let the velocities of A & B be v_1 &v_2 respectively in the same direction.

By Newton's rule, $v_2 - v_1 = \text{-e (O-u)} = \frac{3}{5}u$ (1)

By conservation of momentum along AB,

$30v_2 + 2v_1 = 30 \times 0 + 2u \text{(i.e)} 15v_2 + v_1 = u$ (2)

Solving (1) and (2), we get $v_1 = -\frac{u}{2}$ and $v_2 = \frac{u}{10}$.

Since v_1 is negative the velocity A after the impact **towards the** wall and $= \frac{u}{2}$ while the velocity of B is $\frac{u}{10}$ away from the wall.

Now A strikes the wall with a velocity $\frac{u}{2}$. After this impact, its velocity will be reserved as a e. $\left(\frac{u}{2}\right) = \frac{3}{5}.\frac{u}{2} = \frac{3u}{10}$ with the velocity, A moves in the direction AB, **away from the** wall and strike B a second time. Let the velocities of A and B be v_3 and v_4 **after this impact** in the direction AB.

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For convenience, the velocity distribution can be noted as follows:

	A (2)	B (30)
Before Impact	$\frac{3u}{10}$	$\frac{u}{10}$
After impact	v_3	v_4

By Newton's rule,

$$v_4 - v_3 = -e\left(\frac{u}{10} - \frac{3u}{10}\right) = \frac{3u}{25} \qquad (3)$$

By conservation of momentum,

$$30v_4 + 2v_3 = 30.\frac{u}{10} + 2.\frac{3u}{10} = \frac{18u}{5}$$

$$\text{i.e. } 15v_4 + v_3 = \frac{9u}{5} \qquad (4)$$

Multiplying (3) by 15 we have

$$15v_4 - 15v_3 = \frac{9u}{5}$$

Subtracting (5) from (4), $16v_3 = 0$ or $v_3 = 0$.

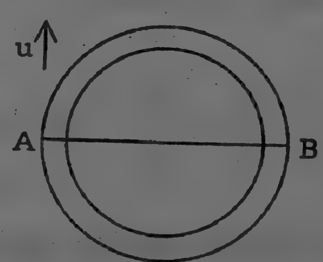
i.e. A is reduced to rest after its second impact with B.

Example: 5

Two equal marble balls A, B lie in a horizontal circular groove at the opposite ends of a diameter; A is projected along the grove and after time t impinges on B; show that a second impact takes place after a further interval $\frac{2t}{e}$.

Solution:

Let the ball A move with velocity u. As there is no tangential force acting on A at any point of its path, its speed remains the same throughout. Hence it impinges on B with a velocity u.



Since the time from A to B is = t.

We get

$ut = \pi r$ or $u = \dfrac{\pi r}{t}$ (1)

Let v and v' be the velocity of A and B respectively after impact.

Then by the principle of momentum,

$mv + mv' = mu$ (m being the mass of each ball)

(i.e.) $v + v' = u$ (2)

Also, be Newton' $v - v' = -e\ (v - o)$

(i.e.) $v - v' = -eu$ (3)

Solving (2) and (3), we get $v = \dfrac{u}{2}$ (i.e); $v' = \dfrac{u}{2}(1+e)$

Clearly v' is greater than v. Hence B will move in advance of A. Let it strike a again t_1 secs. after the first impact.

The velocity of B relative to A, after the first impact

$= v' - v = eu$ from (3)

Before striking again, B should cover a distance equal length to the circumstance relative to A.

$\therefore (v'-v).t_1 = 2\pi r \text{ (or) } eu.t_1 = 2\pi r$

$$t_1 = \frac{2\pi r}{eu} = \frac{2\pi r}{e\left(\dfrac{\pi r}{t}\right)} \text{ using (1)}$$

$$= \frac{2t}{e}$$

$\therefore$ The second impact occurs $\dfrac{2t}{e}$ secs. after the first.

Exercises:

1) Two balls impinge directly and interchange their velocities after impact. Prove that they are perfectly elastic and are of equal masses.

2) A ball A impinges directly on a exactly equal and similar ball B lying on a smooth horizontal plane. If the coefficient of restitution is e, prove that after impact, the velocity of B will be to that of A as 1+e; 1-e.

3) A ball impinges directly on a second ball of twice its mass which is moving in the same direction as the first but with one-seventh of its velocity. Given that co-efficient of restitution is $\dfrac{3}{4}$, show that first ball will come to rest after the impact?

8.2 Oblique impact of two smooth spheres:

Definition: Oblique Impact

Two bodies are said to impinge obliquely if the direction of motion of either or both is not along the common normal at the point of contact.

Result 1:

A smooth sphere of mass m_1 impinge obliquity with velocity u_1 on another smooth sphere of mass m_2 moving with velocity u_2. If the directions of motion before impact make angles α_1 and α_2 respectively with the line joining the centre of the spheres and if the co-efficient of restitution be e, to find the velocities and directions of motion after impact.

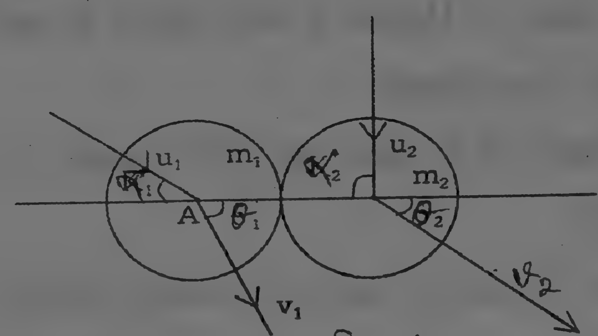


fig.(8.2)

Let the velocities of the spheres after impact be v_1 and v_2 in directions inclined at angle θ_1 and θ_2 respectively to the line of centres. Since the spheres are smooth, there is no force perpendicular to the line of centers and therefore, for each sphere, the velocities in the tangential direction are not affected by impact.

$$\therefore\ v_1\mathrm{Sin}\theta_1 = u_1\mathrm{Sin}\alpha_1 \qquad (1)$$

$$\text{And } v_2\mathrm{Sin}\theta_2 = u_2\mathrm{Sin}\alpha_2 \qquad (2)$$

By Newton's law concerning velocities along the common normal AB,

$$v_2\mathrm{Cos}\theta_2 - v_1\mathrm{Cos}\theta_1$$

$$= \text{-e}\left(u_2\mathrm{Cos}\alpha_2 - u_1\mathrm{Cos}\alpha_1\right) \qquad (3)$$

By the principle of conservation of momentum along AB,

$$m_2v_2\mathrm{Cos}\theta_2 + m_1v_1.\mathrm{Cos}\theta_1 = m_2.u_2\mathrm{Cos}\alpha_2 + m_1.u_1\mathrm{Cos}\alpha_1 \qquad (4)$$

(4) – (3) x m_2 gives

$$v_1\mathrm{Cos}\theta_1(m_1 + m_2) = m_2u_2\mathrm{Cos}\alpha_2 + m_1u_1\mathrm{Cos}\alpha_1 + em_2(u_2\mathrm{Cos}\alpha_2 - u_1\mathrm{Cos}\alpha_1)$$

$$\text{i.e. } v_1\mathrm{Cos}\theta_1 = \frac{u_1\mathrm{Cos}\alpha_1(m_1 - em_2) + m_2u_2\mathrm{Cos}\alpha_2(1+e)}{m_1 + m_2} \qquad (5)$$

(4) + (3) x m_1 gives

$$v_2 Cos\theta_2 = \frac{u_2 Cos\alpha_2 (m_2 - em_1) + m_1 u_1 Cos\alpha_1 (1+e)}{m_1 + m_2} \quad (6)$$

From (1) and (5) by squaring and adding, we obtain v_1^2 and by division, we have $\tan\theta_1$. Similarly from (2) and (6), we get v_2^2 and $\tan\theta_2$. Hence the motion after impact is completely determined.

Corollary: 1

If the two spheres are perfectly elastic and of equal mass, than e = 1 and $m_1 = m_2$.

Then, from equation (5) and (6), we have

$$v_1 Cos\theta_1 = \frac{O + m_1 u_2 Cos\alpha_2 .2}{2m_1} = u_2 Cos\alpha_2$$

and $$v_2 Cos\theta_2 = \frac{O + m_1 u_1 Cos\alpha_1 .2}{2m_1} = u_1 Cos\alpha_1$$

Hence if two equal perfectly elastic spheres impinge, they interchange their velocities in the direction of the line of centres.

Corollary: 2

Usually, in most problems or oblique impact, one of the spheres is at rest. Suppose m_2 is at rest ie, $u_2 = 0$.

From equation (2), $v_2 Sin\theta_2 = 0$ i.e. $\theta_2 = O$.

Hence m_2 moves along AB after impact. This is seen independently, since the only force on m_2 during impact is along the line of centres.

Corollary: 3

The impulse of the blow on the sphere A of mass m_1.

= Change of momentum of A along the common normal

$= m_1 (v_1 Cos\theta_1 - u_1 Cos\alpha_1)$

$$= m_1 \left[\frac{u_1 Cos\alpha_1 (m_1 - em_2) + m_2 u_2 Cos\alpha_1 (1+e)}{m_1 + m_2} - u_1 Cos\alpha_1 \right]$$

$$= \frac{m_1 \begin{bmatrix} m_1 u_1 Cos\alpha_1 - em_2 u_1 Cos\alpha_1 + m_2 u_2 Cos\alpha_2 + em_2 u_2 Cos\alpha_2 \\ - m_1 u_1 Cos\alpha_1 - m_2 u_1 Cos\alpha_1 \end{bmatrix}}{m_1 + m_2}$$

$$= \frac{m_1 \lfloor m_2 u_2 Cos\alpha_2 (1+e) - m_2 u_1 Cos\alpha_1 (1+e) \rfloor}{m_1 + m_2}$$

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$$= \frac{m_1 m_2 (1+e)}{m_1 + m_2}(u_2 \cos\alpha_2 - u_1 \cos\alpha_1)$$

The impulsive blow on m_2 will be equal and opposite of the impulsive blow on m_1.

8.2.1 Loss of Kinetic energy due to oblique impact of two smooth spheres:

Two spheres of masses m_1 and m_2 moving with velocities u_1 and u_2 at angles α_1 and α_2 with their line of centers come into collision. To find an expression for the loss of kinetic energy.

The velocities perpendicular to the line of centres are not altered by impact. Hence the loss of kinetic energy in the case of oblique impact is therefore the same as in the case of direct impact if we replace in the expression (4) on page No.256 the quantities u_1 and u_2 by $u_1 \cos\alpha_1$ and $u_2 \cos\alpha_2$ respectively.

Therefore the loss is $\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(1-e^2)(u_1 \cos\alpha_1 - u_2 \cos\alpha_2)^2$

We shall now derive this independently. Let v_1 and v_2 by the velocities of the sphere after impact, in directions inclined at angles θ_1 and θ_2 respectively to the line of centres. As explained in the tangential velocity of each sphere is not altered by impact.

$\therefore v_1 \sin\theta_1 = u_1 \sin\alpha_1 .....(1)$ and $v_2 \sin\theta_2 = u_2 \sin\alpha_2$ (2)

By Newton's rule

$$v_2 \cos\theta_2 - v_1 \cos\theta_1 = -e(u_2 \cos\alpha_2 - u_1 \cos\alpha_1) \quad (3)$$

By conservation of momenta

$$m_2 v_2 \cos\theta_2 + m_1 v_1 \cos\theta_1 = m_2 u_2 \cos\alpha_2 + m_1 u_1 \cos\alpha_1$$

i.e. $m_1 (u_1 \cos\alpha_1 - v_1 \cos\theta_1) = m_2 (v_2 \cos\theta_2 - u_2 \cos\theta_2)$ (4)

Change in K.E.

$$= \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 - \frac{1}{2}m_1 v_1^2 - \frac{1}{2}m_2 v_2^2$$

$$= \frac{1}{2}m_1 u_1^2(\cos^2\alpha_1 + \sin^2\alpha_1) + \frac{1}{2}m_2 u_2^2(\cos^2\alpha_2 + \sin^2\alpha_2)$$

$$- \frac{1}{2}m_1 v_1^2(\cos^2\theta + \sin^2\theta_1) - \frac{1}{2}m_2 v_2^2(\cos^2\theta_2 + \sin^2\theta_2)$$

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$$=\frac{1}{2}m_1u_1^2Cos^2\alpha_1+\frac{1}{2}m_2u_2^2Cos\alpha_2-\frac{1}{2}m_1v_1^2Cos^2\theta_1-\frac{1}{2}m_2u_2^2Cos^2\theta_2$$

(Using (1) and (2))

$$=\frac{1}{2}m_1\left(u_1^2Cos^2\alpha_1-v_1^2Cos^2\theta_1\right)+\frac{1}{2}m_2\left(u_2^2Cos^2\alpha_2-v_2^2Cos^2\theta_2\right)$$

$$=\frac{1}{2}m_1\left(u_1Cos\alpha_1+v_1Cos\theta_1\right)\left(u_1Cos\alpha_1-v_1Cos\theta_1\right)+$$

$$\frac{1}{2}\left(u_2Cos\alpha_2+v_2Cos\theta_2\right).\left(u_2Cos\alpha_1-v_2Cos\theta_2\right)$$

$$=\frac{1}{2}m_1\left(u_1Cos\alpha_1+v_1Cos\theta_1\right)\left[u_1Cos\alpha_1-v_1\cos\theta_1\right]-\frac{1}{2}\left(u_2Cos\alpha_2+v_2Cos\theta_2\right)$$

$$m_1\left(u_1Cos\alpha_1-v_1Cos\theta_1\right)$$

using (4)

$$=\frac{1}{2}m_1(u_1\cos\alpha_1-v_1\cos\theta_1)(u_1\cos\alpha_1+v_1\cos\theta_1-u_2\cos\alpha_2-v_2\cos\theta_2)$$

$$=\frac{1}{2}m_1\left(u_1Cos\alpha_1-v_1Cos\theta_1\right)\left[u_1Cos\alpha_1+u_2Cos\alpha_2\quad e\left(u_2Cos\alpha_2-u_1Cos\alpha_1\right)\right]$$

Using (3)

$$=\frac{1}{2}m_1\left(u_1Cos\alpha_1-v_1Cos\theta_1\right)\left(u_1Cos\alpha_1-u_2Cos\alpha_2\right)(1-e) \qquad (5)$$

Now from

$$\frac{u_1Cos\alpha_1-v_1Cos\theta_1}{m_2}=\frac{v_2Cos\theta_2-u_2Cos\alpha_2}{m_1}$$

And each $=\dfrac{u_1Cos\alpha_1-v_1Cos\theta_1+v_2Cos\theta_2-u_2Cos\alpha_2}{m_1+m_2}$

$$=\frac{\left(u_1Cos\alpha_1-u_2Cos\theta\alpha_2+\left(v_2Cos\theta_2-v_1Cos\theta_1\right)\right)}{m_1+m_2}$$

$$=\frac{u_1Cos\alpha_1-u_2Cos\alpha_2-e\left(u_2Cos\alpha_2-u_1Cos\alpha_1\right)}{m_1+m_2}\text{ Using (3)}$$

$$=\frac{\left(u_1Cos\alpha_1-u_2Cos\alpha_2\right)(1+e)}{m_1+m_2}$$

$$\therefore u_1Cos\alpha_1-v_1Cos\theta_1=\frac{m_2(1+e)}{m_1+m_2}\left(u_1Cos\alpha_1-u_2Cos\alpha_2\right)$$

Substituting in (5),

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Change in K.E. =

$$\frac{1}{2}\frac{m_1m_2(1+e)}{m_1+m_2}(u_1Cos\alpha_1 - u_2Cos\alpha_2)x(u_1Cos\alpha_1 - u_2Cos\alpha_2)(1-e)$$

$$=\frac{1}{2}\frac{m_1m_2}{m_1+m_2}(1+e^2)(u_1Cos\alpha_1 - u_2Cos\alpha_2)^2$$

If the spheres are perfectly elastic, e = 1 and the loss of kinetic energy is zero.

8.2.2 Dissipation of energy due to impact:

We have found that in any impact, except where the coefficient of restitution of unity, some kinetic energy is lost. This missing kinetic energy is converted into other forms of energy and chiefly reappears in the shape of heat. Hence the principle of conservation of energy will not hold good in problems of impact.

Example: 1

A ball of mass 8 gms. moving with velocity 4 cms. per sec. impinges on a ball of mass 4 gms. moving with velocity 2cm. per sec. If their velocities before impact be inclined at angel 30° and 60° to the line joining their centres at the moment of impact find their velocities after impact when $e=\frac{1}{2}$.

Refer to oblique impact, $m_1 = 8$, $u_1 = 4$; $\alpha_1 = 30°$, $m_2 = 4$; $u_2 = 2$; $\alpha_2 = 60°$ Let v_1 and v_2 be the velocities after impact in directions making θ_1 and θ_2 respectively with AB. The tangential velocity of each sphere is not affected by impact.

$\therefore v_1Sin\theta_1 = 4\,Sin\,30° = 2$ (1)

And $v_2Sin\theta_2 = 2\ Sin\,60° = \sqrt{3}$ (2)

By Newton's law,

$$v_2Cos\theta_2 - v_1Cos\theta_1 = -e(2Cos60° - 4Cos30°)$$

$$=-\frac{1}{2}\left(2.\frac{1}{2} - 4.\frac{\sqrt{3}}{2}\right)$$

$$=\frac{1}{2}(2\sqrt{3}-1) \qquad (3)$$

By conservation of momenta along AB.

$$4v_2Cos\theta_2 + 8v_1Cos\theta_1 = 4.2Cos60° + 8.4Cos30°$$

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$= 4 + 16\sqrt{3}$

i.e. $v_2 \text{Cos}\theta_2 + 2v_1 \text{Cos}\theta_1 = 1 + 4\sqrt{3}$ (4)

$\therefore\ 3V_1 \text{Cos}\theta_1 = 1 + 4\sqrt{3} - \frac{1}{2}\left(2\sqrt{3} - 1\right) = \frac{3 + 6\sqrt{3}}{2}$

i.e. $v_1 \text{Cos}\theta_1 = \frac{1 + 2\sqrt{3}}{2}$ (5)

From (4), $v_2 \text{Cos}\theta_2 = 1 + 4\sqrt{3} - 1 - 2\sqrt{3} = 2\sqrt{3}$ (6)

From (1) and (5), $v_1^2 = 2^2 + \left(\frac{1 + 2\sqrt{3}}{2}\right)^2$

$= 4 + \frac{1 + 4\sqrt{3} + 12}{4} = \frac{29 + 4\sqrt{3}}{4}$

$\therefore v_1 = \sqrt{\frac{29 + 4\sqrt{3}}{2}}$ cm. per sec.

Dividing (1) by (5)

$\tan\theta_1 = \frac{4}{1 + 2\sqrt{3}}$

From (2) and (6)

$v_2^2 = 3 + 12 = 15$ and

$\therefore v_2 = \sqrt{15}$ cm/sec.

Dividing (2) by (6)

$\tan\theta_2 = \frac{1}{2}$

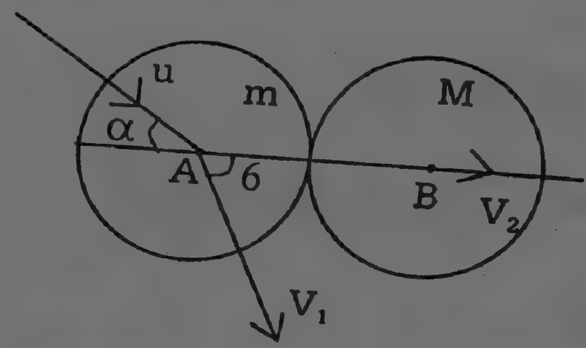
Example 2:

A smooth sphere of mass m impinges obliquely on a smooth sphere of mass M which is at rest. Show that if m = eM, the directions of motion after impact are at right angles. (e is the coefficient of restitution)

Solution:

Considering the sphere M, its tangential velocity before impact is zero and hence after impact also, its tangential velocity is zero. ($\because$ During impact, there is no force acting along the common tangent). Hence after impact, M will move along AB. Let its velocity be v_2. Let the velocity of m be V at an angle θ to AB, after impact.

Space for Hints



By Newton's rule,

$$v_2 - v_1 \text{Cos}\theta = -e(O - u\,\text{Cos}\alpha)$$

(i.e) $v_2 - v_1\text{Cos}\theta = eu\,\text{Cos}\alpha$ (1)

By conservation of momenta along AB,

$M.V_2 + mV_1\,\text{Cos}\theta = M.O + m.u\,\text{Cos}\alpha$ (2)

Multiplying (1) by M and subtracting from (2),

$$mv_1\,\text{Cos}\theta + Mv_1\text{Cos}\theta = mu\,\text{Cos}\alpha - Meu\,\text{Cos}\alpha$$

(i.e) $v_1\text{Cos}\theta = \dfrac{u\,\text{Cos}\alpha(m - eM)}{m+M} = \dfrac{u\,\text{Cos}\alpha.O}{m+M}(\because m = eM)$

$= 0.$

$\therefore \text{Cos}\theta = 0$ or $\theta = 90^\circ$

Example: 3

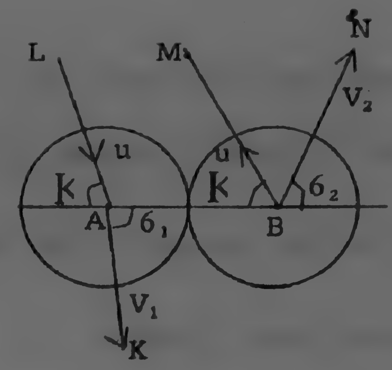
Two equal elastic balls moving in opposite parallel direction with equal speeds impinge on one another. If the application of their direction of motion to the line of centres be $\tan^{-1}\left(\sqrt{e}\right)$ where e is the coefficient of restitution, show that their direction of motion will be turned through a right angle.

Solution:

Let m be the mass of either sphere. AB is the line of impact.

Before impact, the directions of motion are LA and BM making the same acute angle α with AB as shown is the figure. Let u be their velocity.

After impact, let the sphere A proceed in the direction AK with velocity v_1 at an angle θ_1, to AB and the sphere B proceed in the direction BN with velocity v_2 at angle θ_2 to AB.



The tangential velocity of either sphere is not affected by impact

$$\therefore v_1 \operatorname{Sin}\theta_1 = u \operatorname{Sin}\alpha \quad (1)$$

$$v_2 \operatorname{Sin}\theta_2 = u \operatorname{Sin}\alpha \quad (2)$$

By Newton's law, (resolving all velocities along AB)

$$v_2 \operatorname{Cos}\theta_2 - v_1 \operatorname{Cos}\theta_1 = -e(-u \operatorname{Cos}\alpha - u \operatorname{Cos}\alpha)$$

(i.e.) $v_2 \operatorname{Cos}\theta_2 - v_1 \operatorname{Cos}\beta\theta_1 = 2eu \operatorname{Cos}\alpha$ (3)

By conservation of momenta along AB

$$m(v_2\operatorname{Cos}\theta_2) + m.v_1 \operatorname{Cos}\theta_1 = m(-u \operatorname{Cos}\alpha) + mu \operatorname{Cos}\alpha$$

(i.e) $v_2\operatorname{Cos}\theta_2 + v_1 \operatorname{Cos}\theta_1 = 0$ (4)

(4) – (3) gives $2v_1 \operatorname{Cos}\theta_1 = -2 eu \operatorname{Cos}\alpha$

$$\therefore v_1\operatorname{Cos}\theta_1 = -eu \operatorname{Cos}\alpha \quad (5)$$

From (4) $v_2\operatorname{Cos}\theta_2 = -v_1 \operatorname{Cos}\theta_1 = eu \operatorname{Cos}\alpha$ (6)

Dividing (1) by (5)

$$\tan\theta_1 = -\frac{1}{e}\tan\alpha = -\frac{1}{6}\sqrt{e}(\because \alpha = \tan^{-1}\sqrt{e} \text{ given}$$

$$= -\frac{1}{\sqrt{e}} = -\frac{1}{\tan\alpha} = -\operatorname{Cot}\alpha$$

$$= \tan(90° + \alpha)$$

$$\therefore \theta_1 = 90° + \alpha$$

Dividing (2) by (6) $\tan\theta_2 = \frac{1}{e}\tan\alpha = \operatorname{Cot}\alpha = \tan(90° - \alpha)$

$$\therefore \theta_2 = 90° - \alpha.$$

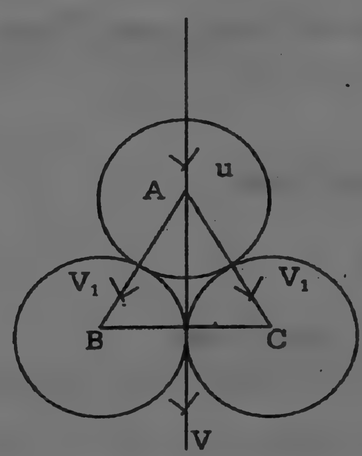
Hence their directions of motion are turned through a right angle.

Example: 4

Two equal billiard balls are in contact on a smooth table and a third equal ball, moving along their common tangent, strikes them simultaneously. Prove that $\frac{3}{5}(1-e^2)$ of its kinetic energy is lost by the impact, e bring the coefficient of restitution for each pair of balls.

Solution:

Let A be the centre of the impinging ball, and B,C the centres of the other two balls at rest. Since the balls are equal, Δ ABC is equilateral. Let u be the velocity of A before impact. After impact let v be the velocity of A and v_1 the common velocity of B and C. v will be in the same line as u, while B and C, will move along AB and AC respectively. Let m be the mass of each ball. We apply the principle of conservation of momentum for all the 3 balls along the direction of motion of A.



$\therefore mv + 2m.v_1 \cos 30° = mu$

(i.e.) $v + v_1\sqrt{3} = u$ (1)

Applying Newton's Law along AB for the spheres A and B.

We have

$v_1 - v\cos 30° = -e(0 - u\cos 30°)$

(i.e) $v_1 - \frac{v\sqrt{3}}{2} = \frac{eu\sqrt{3}}{2}$

or $-v\sqrt{3} + 2v_1 = eu\sqrt{3}$ (2)

Multiplying (1) by $\sqrt{3}$ and adding to (2)

We have

$$5v_1 = u\sqrt{3}(1+e) \text{ or } v_1 = \frac{u\sqrt{3}(1+e)}{5} \qquad (3)$$

From (1), $v = u - v_1\sqrt{3} = u - \frac{3u(1+e)}{5} = \frac{u(2-3e)}{5}$ (4)

The loss of kinetic energy,

= Initial K.E – Final K.E.

$$= \frac{1}{2}mu^2 - \left(\frac{1}{2}mv^2 + \frac{1}{2}mv_1^2 + \frac{1}{2}mV_1^2\right)$$

$$= \frac{1}{2}mu^2 - \frac{1}{2}mv^2 - mv_1^2$$

$$= \frac{1}{2}mu^2 - \frac{1}{2}mu^2.\frac{(2-3e)^2}{25} - \frac{m.3u^2(1+e)^2}{25}$$

$$= \frac{1}{2}mu^2\left[1 - \frac{(4-12e+9e^2)}{25} - \frac{6}{25}(1+2e+e^2)\right]$$

$$= \frac{1}{2}mu^2\left[\frac{25-4+12e-9e^2-6-12e-6e^2}{25}\right]$$

$$= \frac{1}{2}mu^2.\frac{(15-15e^2)}{25} = \frac{1}{2}mu^2.\frac{3}{5}(1-e^2)$$

$= \frac{3}{5}(1-e^2)$ x the original K.E.

Exercises:

1) A sphere of mass m moving on a horizontal plane with velocity impinges obliquely on a sphere of mass m' at rest on the same plane. If e = 1 and m = m', prove that the directions of motion after impact are at right angles.

2) If two equal perfectly elastic spheres impinge obliquely, prove that they interchange their velocities in the direction of the line of centres.

3) Two equal perfectly elastic balls impinge if their directions of motion before impact be at right angles, show that their directions of motion after impact are also at right angles.

UNIT – 9

SIMPLE HARMONIC MOTION

9.1 Introduction

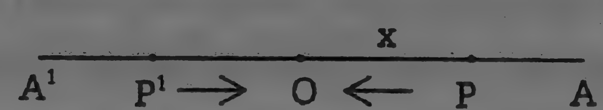
A very common and important type of motion occurring in nature is that which involves oscillations backwards and forwards about some fixed point. For instance, suppose one end of an elastic string is tied to a fixed point and a heavy particle is attached to the other end. If the particle is disturbed vertically from its position of equilibrium, it is found that it oscillates to and from about this position. Clearly the particle cannot be moving under constant acceleration. It is found that it has an acceleration, which is always directed towards the equilibrium position and varies in magnitude as the distance of the particle from that position. This kind of motion occurs frequently in nature and since it is of the type which produces all musical notes. It is called Simple Harmonic Motion (S.H.M). The oscillations of a simple pendulum and the transverse vibrations of a plucked violin string are examples of Simple Harmonic Motion.

9.2 Simple Harmonic Motion in a Straight Line:

Definition

When a particle moves in a straight line so that its acceleration is always directed towards a fixed point in the line and proportional to the distance from that point, its motion is called Simple Harmonic Motion.

Let O be a fixed point on the straight line $A'OA$ on which a particle is having Simple Harmonic Motion. Take O as the Origin and OA as the x-axis.



Let P be the position of the particle at time t. Such that OP = x. The magnitude of the acceleration at P = μx where μ is a positive constant. As this acceleration acts towards O, the acceleration at P in the positive direction of the x-axis is $-\mu x$.

Hence the equation of motion of P is

$$\frac{d^2x}{dt^2} = -\mu x \qquad (1)$$

Here it must be noted that for a position of P to the right of O, the x-coordinate x is positive and so the acceleration $\frac{d^2x}{dt^2}$ is negative directed towards O. If P' is a position of the particle to the left of O, x is negative and so the acceleration $\frac{d^2x}{dt^2}$ is positive again towards O.

Hence the same equation of motion (1) holds good for all positions of P on the line.

Equation (1) is the fundamental differential equation representing a S. H. M. we now proceed to solve it

If v is the velocity of the particle at time t.

$$1 \Rightarrow \frac{d^2x}{dt^2} = -\mu x$$

Put $\frac{d^2x}{dt^2} = \frac{v.dv}{dx}$

$$\therefore v.\frac{dv}{dx} = -\mu x$$

$$v\,dv = -\mu X dx \qquad (2)$$

Integrating (2)

We have $\frac{v^2}{2} = -\frac{\mu X^2}{2} + c \qquad (3)$

Where c is the constant of integration.

Initially let the particle start from rest at the point A where OA = a and let us measure time also from this instant.

Hence when x = a, v = 0.

Putting these in (3)

$$O = -\frac{\mu a^2}{2} + c \text{ (or) } c = \frac{\mu a^2}{2}$$

$$v^2 = -\mu x^2 + 2c$$

$$v^2 = -\mu x^2 + \frac{2\mu a^2}{2}$$

$$v^2 = -\mu x^2 + \mu a^2 = \mu(a^2 - x^2) \qquad (4)$$

$$\therefore v = \pm\sqrt{\mu(a^2 - x^2)}$$

Space for Hints

Equation (4) gives the velocity v corresponding to any displacement x.

Now, as t increases, x decreases.

So $\frac{dx}{dt} = v = -\sqrt{\mu(a^2 - x^2)}$

(or) $\frac{-dx}{\sqrt{a^2 - x^2}} = \sqrt{\mu}dt$.

Integrating $\text{Cos}^{-1}x/a = \sqrt{\mu}t + A$

Initially when $t = 0$, $x = a$.

$\therefore \text{Cos}^{-1}1 = 0 + A$

(i.e) $A = 0$.

Hence $\text{Cos}^{-1}\frac{x}{a} = \sqrt{\mu}t$

$\therefore \frac{x}{a} = \text{Cos}\sqrt{u}t$

$\therefore x = a\ \text{Cos}\sqrt{\mu}t$ (6)

Equation (6) gives the displacement x in terms of time t.

When the particle comes to O, x = 0 and by (5)

Its velocity then = $-a\sqrt{\mu}$

So the particle passes through O and immediately the acceleration alters its direction and tends to decreases the velocity.

From (5) v = 0 when x = -a.

So the particle comes to rest at a point A' to the left of O such that $OA = OA'$, If then retraces its path passes through O, an again is instantaneously at rest at A. The whole motion of the particle is an oscillation from A to A' and back.

To get the time from A to A'.

Put x = -a in (6)

We have $\text{Cos}\ \sqrt{\mu}t = -1 = \text{Cos}\pi$

$\therefore t = \frac{\pi}{\sqrt{\mu}}$

The time from A to A' and back = $\frac{2\pi}{\sqrt{\mu}}$

We have equation (6) can be written as

$$X = a\,Cos\sqrt{\mu}\,t = a\,Cos\left(\sqrt{\mu}t + 2\pi\right)$$

$$= a\,Cos\left(\sqrt{\mu}t + 4\pi\right) \text{etc.}$$

$$= a\,Cos\left(\sqrt{\mu}\left(t + \frac{2\pi}{\sqrt{\mu}}\right)\right)$$

$$= a\,Cos\sqrt{\mu}\left(t + \frac{4\pi}{\sqrt{\mu}}\right) \text{etc.}$$

This shows that the displacement of the particle at any particular time t, in repeated at times $t_2 + \frac{2\pi}{\sqrt{\mu}}.t_1 + \frac{4\pi}{\sqrt{\mu}}$ etc.

Differentiating (6)

$$\frac{dx}{dt} = -a\sqrt{\mu}\,Sin\sqrt{\mu}t$$

$$= -a\sqrt{\mu}\,Sin\left(\sqrt{\mu}t + 2\pi\right)$$

$$= -a\sqrt{\mu}\,Sin\left(\sqrt{\mu}t + 4\pi\right) \text{etc.}$$

$$= -a\sqrt{\mu}\,Sin\sqrt{\mu}\left(t + \frac{2\pi}{\sqrt{\mu}}\right)$$

$$= -a\sqrt{\mu}\,Sin\sqrt{\mu}\left(t + \frac{4\pi}{\sqrt{\mu}}\right) \text{etc.}$$

This shows that the values of $\frac{dx}{dt}$ are the same if t is increased by $\frac{2\pi}{\sqrt{\mu}}$ or by any multiple of $\frac{2\pi}{\sqrt{\mu}}$. Hence after a time $\frac{2\pi}{\sqrt{\mu}}$ the particle is again at the same particular point moving with the same velocity in the same direction as before, having covered the whole path of the motion just once. The particle is said to be the period $\frac{2\pi}{\sqrt{\mu}}$.

9.2.1 Definitions:

The period or the periodic time of a simple harmonic motion is the internal of time that elapses from any instant till a subsequent instant when the particle is again moving through the same position with the same velocity in the same direction. The frequency of the oscillation is the number of complete oscillations

Space for Hints

that the particle makes in one second. So frequency is the reciprocal of the period and in equal to $\frac{\sqrt{\mu}}{2\pi}$.

The distance through which the particle moves away from the centre of motion on either side of it is called the amplitude of the oscillation.

Thus in the above case amplitude = OA = OA' = a.

We notice that the periodic time being = $\frac{2\pi}{\sqrt{\mu}}$ is independent of the amplitude which is the distance from the centre at which the particle started. It depends only on the constant μ. Which is the acceleration at unit distance from the centre.

Note:

1) Since $\frac{d^2x}{dt^2} = -\mu x$, maximum acceleration corresponds to the greatest value of x and so it is numerically = $\mu.a = \mu$ (Amplitude)

2) Since $v = \sqrt{\mu(a^2 - x^2)}$, the greatest value of v is got at x = 0 and it is $= a\sqrt{\mu} = \mu$ (amplitude)

9.2.2 General Solution of the S. H. M. equation

The S. H. M. equation is $\frac{d^2x}{dt^2} = -\mu x$

$$\text{(i.e.) } \frac{d^2x}{dt^2} + \mu x = 0 \qquad (1)$$

(1) is a linear differential equation of the second order with constant coefficients. Its most general solution is of the form

$$x = A\,\mathrm{Cos}\sqrt{\mu}\,t + B\,\mathrm{Sin}\sqrt{\mu}\,t \qquad (2)$$

Where A and B are arbitrary constants.

Other forms of the solution equivalent to (2) are $x = C\ \mathrm{Cos}\left(\sqrt{\mu}\,t + \varepsilon\right)$ (3)

And $x = D\ \mathrm{Sin}\left(\sqrt{\mu}\,t + \alpha\right)$ (4)

The constants A and B in (2), C and E in (3) and D and α in (4) are known if we know the values of x and $\frac{dx}{dt}$ corresponding to a given time t.

From (3) and (4), the maximum value of x = C or D.

Hence if a is the amplitude of the motion, the forms (3) and (4) can be respectively put as.

$$X = a\,Cos\left(\sqrt{\mu}\,t+\varepsilon\right) \tag{5}$$

$$X = a\,Sin\left(\sqrt{\mu}\,t+\alpha\right) \tag{6}$$

When the solution of the S. H. M equation is expressed as $X = a\,Cos\left(\sqrt{\mu}\,t+\varepsilon\right)$, the quantity E is called the epoch. The phase of a S. H. M at any instant is the time that has clasped since the particle was at its maximum distance in the positive direction.

From the equation (5), x is maximum when $Cos\left(\sqrt{\mu}\,t+\varepsilon\right)=1$. If t_o is the value of t,

$$\sqrt{\mu}t_o+\varepsilon=0$$

(i.e.) $t_o=-\dfrac{\varepsilon}{\sqrt{\mu}}$

Hence phase at time $t = t - t_o$

$$= t+\frac{\varepsilon}{\sqrt{\mu}}$$

$$= \frac{\sqrt{\mu}t+\varepsilon}{\sqrt{\mu}}$$

Note:

Two simple harmonic motions of the same period can be represented by

$$X_2 = a_1 Cos\left(\sqrt{\mu}t+\varepsilon_1\right)$$

$$X_2 = a_2 Cos\left(\sqrt{\mu}t+\varepsilon_2\right)$$

Then difference in phase $= \dfrac{\varepsilon_1-\varepsilon_2}{\sqrt{\mu}}$

If $\varepsilon_1=\varepsilon_2$, the motion, are in the same phases.

If $\varepsilon_1=\varepsilon_2$, they are in opposite phases.

9.2.3 Geometrical Representation of a Simple Harmonic Motion:

Let a point Q describe with uniform angular velocity w, a circle of radius a and centre O of which A'OA is fixed diameter. Let P be the foot of the perpendicular from O on AA'.

As O moves round the circle, P will moves to and form on the diameter.

Space for Hints

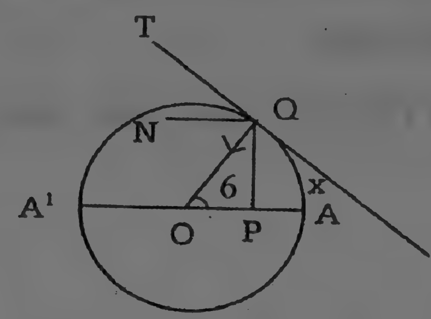
Space for Hints

We can show that the motion of P along AA' is simple harmonic, with O at centre.

Let Q move in the direction AOA' as shown in the figure. As Q moves uniformly in a circle, its only acceleration is $\omega^2.QO$ along QO.

The velocity of Q in the circle is $\omega.QO$ along the tangent QT.

The velocity and acceleration of P must be the same as the resolved parts along AA', of the velocity and acceleration of Q.



Hence acceleration of $P = \omega^2 . QO. \cos \angle POQ$

$$= \omega^2 . QO. \frac{PO}{QO}$$

$$= \omega^2 . PO \text{ towards O.}$$

(i.e.) the acceleration of P is always directed towards O and proportional to its distance from O. Hence the motion of P is simple harmonic.

The various formulae of S. H. M derived can be deduced by considering the motion of Q along the circle.

Taking O as the origin and OA as the positive direction of measuring displacement. Let $OP = X$ and $\angle QOP = \theta$

Velocity of Q = aw along QT.

Hence velocity of P = resolved part of velocity of Q along AA'.

$$= a\omega \cos\angle TQN = a\omega \sin\angle OQN$$

$$= a\omega \sin\theta = a\omega.\frac{PQ}{a}$$

$$= \omega\sqrt{OQ^2 - OP^2}$$

$$= \omega\sqrt{a^2 - x^2} \text{ (in magnitude)} \qquad (1)$$

And this velocity of P is along AO towards O.

As Q moves round the circle from A to A' and back to A, P moves from A to A' through O and back A.

Hence the periodic time of the S. H. M. described by P

= Time taken for Q to describe the circle

$$= \frac{2\pi}{\omega} \qquad (2)$$

Also, if t is the time from A to Q

$$t = \frac{\theta}{\omega} = \frac{\operatorname{Cos}^{-1}\left(\frac{x}{a}\right)}{\omega}$$

$$\omega t = \operatorname{Cos}^{-1}\left(\frac{x}{a}\right) \text{ or } x = a \operatorname{Cos} \omega t \qquad (3)$$

The acceleration of P towards O $= \omega^2$. PO and putting $\omega^2 = \sqrt{\mu}$.

(i.e.) $\omega = \sqrt{\mu}$ in (1), (2), (3)

We get

(i) The velocity of P = $\sqrt{\mu}.\sqrt{a^2 - x^2}$ in magnitude

(ii) The periodic time of $P = \frac{2\pi}{\sqrt{\mu}}$

(iii) The displacement $x = a \operatorname{Cos}\sqrt{\mu}\, t$

These are the formulae derived in section 9.2

9.2.4 Change of origin:

A differential equation of the form $\frac{d^2x}{dt^2} = -\mu x$ where μ is a positive number, always represents a simple harmonic motion of period $\frac{2\pi}{\sqrt{\mu}}$ which is independent of the amplitude. The centre of the S. H. M. is the origin from where the displacement x is measured.

Consider now the equation,

$$\frac{d^2x}{dt^2} = -\mu x + \alpha \qquad (1)$$

This can be written as $\frac{d^2x}{dt^2} = -\mu\left(x - \frac{\alpha}{\mu}\right)$ (2)

Put $x - \frac{\alpha}{\mu} = x$ (3)

Space for Hints

When $x = \frac{\alpha}{\mu}, x = 0$

So this means that we are transferring the origin for measuring displacement, to the point distance $\frac{\alpha}{\mu}$ from the original origin.

Differentiating (3) twice

$$\frac{d^2x}{dt^2} = \frac{d^2x}{dt^2} \text{ and hence (2) becomes } \frac{d^2x}{dt^2} = -\mu x \qquad (4)$$

(4) clearly represents a simple harmonic motion about the new origin.

Worked examples

Example: 1

A particle is moving with S. H. M and while making an oscillation from one extreme position to the other, its distance from the centre of oscillation at 3 consecutive seconds are x_1, x_2, x_3. Prove that the period of oscillation is $\dfrac{2\pi}{\cos^{-1}\left(\dfrac{x_1 + x_3}{2\pi}\right)}$.

Solution:

If a is the amplitude, the constant of the S.H.M and x is the displacement at time t, we know that $x = a \cos \sqrt{\mu}t$.

Let at three consective seconds $t_1, t_1 + 1, t_1 + 2$ the corresponding displacement be x_1, x_2, x3.

Then $x_1 = a \cos\sqrt{\mu}\, t_1$

$X_2 = a \cos\sqrt{\mu}(t_1 + 1) = a \cos\left(\sqrt{\mu}t_1 + \sqrt{\mu}\right)$

And $X_3 = a \cos\sqrt{\mu}\,(t_1 + 2) = a \cos\left(\sqrt{\mu}t_1 + 2\sqrt{\mu}\right)$

$\therefore x_1 + x_2 = a[\cos(\sqrt{\mu}t_1 + 2\sqrt{\mu}) + \cos\sqrt{\mu}t_1]$

$= a.2 \cos\dfrac{\sqrt{\mu}t_1 + 2\sqrt{\mu} + \sqrt{\mu}t_1}{2} \cos\dfrac{\sqrt{\mu}t_1 + 2\sqrt{\mu} - \sqrt{\mu}t_1}{2}$

$= 2a \cos\left(\sqrt{\mu}t_1 + \sqrt{\mu}\right)\cos\sqrt{\mu}$

$= 2x_2\cos\sqrt{\mu}$

$$\frac{x_1 + x_2}{2x_2} = \text{Cos}\sqrt{\mu} \quad \text{or} \quad \sqrt{\mu} = \text{Cos}^{-1}\left(\frac{x_1 + x_2}{2 \text{x} 2}\right)$$

$$\therefore \text{Period} = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\text{Cos}^{-1}\left(\frac{x_1 + x_3}{2x_2}\right)}$$

Example: 2

If the displacement of a moving point at any time given by an equation of the form $x = a \text{ Cos}\,\omega t + b \text{ Sin}\,\omega t$. Show that the motion is a simple harmonic motion.

If a = 3, b = 4, w = 2 determine the period amplitude, maximum velocity and maximum acceleration of the motion.

Solution:

$$x = a\cos\omega t + b\sin\omega t \qquad (1)$$

We have to show that the acceleration varies directly as the displacement.

Differentiating (1) w.r to t.

$$\frac{dx}{dt} = -a\omega \text{Sin}\omega t + b\omega \text{Cos}\omega t \qquad (2)$$

$$\frac{d^2x}{dt^2} = a\omega^2 \text{Cos}\omega t - b\omega^2 \text{Sin}\omega t$$

$$= -\omega^2[a\text{Cos}\omega t + b\text{Sin}\omega t] = -\omega^2 x \qquad (3)$$

(3) Shows that the motion is simple harmonic

The constant μ of the S.H.M. = ω^2

$$\therefore \text{ Period } = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi \text{Secs}$$

Amplitude is greatest value of x.

When x is maximum $\frac{dx}{dt} = 0$

$$\therefore -a\omega\text{Sin}\omega t + b\omega\text{Cos}\omega t = 0$$

(i.e.) $a\,\text{Sin}\omega t = b\,\text{Cos}\omega t$

or $\tan\omega t = \frac{b}{a} = \frac{4}{3}$ using the given values. When $\tan\omega t = \frac{4}{3}$,

$$\text{Sin}\omega t = \frac{4}{5} \text{and} \text{Cos}\omega t = \frac{3}{5}$$

putting these values in (1), greatest value of x.

$$= a \times \frac{3}{5} + b \times \frac{4}{5} = \frac{3a+4b}{5} = \frac{3.3+4.4}{5} = 5$$

$\therefore$ Amplitude = 5.

Using the formulae

Max. Acceleration $= \mu$. Amplitude = 4 x 5 = 20

Max. Velocity $= \sqrt{\mu}$. Amplitude = 2 x 5 = 10

Example: 3

A horizontal shelf moves vertically with S.H.M whose complete period is one second; find the greatest amplitude in centimeters. It can have, so that an object resting on the shelf may always remain in contact.

Solutions:

Let m be the mass of an object lying on the shelf, O the centre of the S.H.M and P the position of m at time t.

Let OP = x.

The forces acting on the mass at P are

i) Its weight mg acting vertically downwards and

ii) The nominal reaction R due to the shelf acting upwards.

Resultant force on the mass = mg – R and

So the acceleration on it = $\frac{mg - R}{m}$

And this acts towards O.

Since the particle is moving with S.H.M towards O, Acceleration at $P = \mu.PO = \mu x$.

$$\therefore \mu x = \frac{mg - R}{m}$$

(ie) $R = mg - m\mu x = m\,(g - \mu x)$

Period of the S.H.M $= \frac{2\pi}{\sqrt{\mu}} = 1$ (given)

$\therefore \sqrt{\mu} = 2\pi$ (or) $\mu = 4\pi^2$

$\therefore$ From (1), $R = m\,(g - 4\,\pi^2 x)$

For the mass to remain always in contact with the shelf, **reaction R** must not be negative.

$\therefore m(g - 4\pi^2 x) \geq 0$

(ie) $(g - 4\pi^2 x) \geq 0$ or $x \leq \dfrac{g}{4\pi^2}$

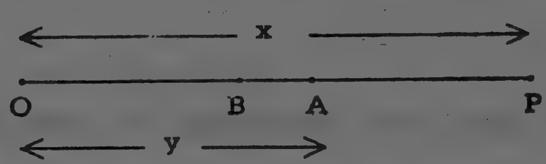
$\therefore$ Greatest value of $x = \dfrac{g}{4\pi^2} = \dfrac{981}{4\pi^2}$ =24.8 cms

Example: 4

A particle P, of mass m, moves in a straight line OX under a force $m\mu$ (distance) directed towards a point A which moves in the straight line OX with constant acceleration α. Show that the motion of P is simple harmonic of period $2\pi/\sqrt{\mu}$ about a moving centre which is always at a distance $a/\sqrt{\mu}$ behind A.

Solution:

Let at time t, the particle be at P where OP = x.



And A be such that OA = y.

The equation of motion of P is

$$\frac{d^2}{dt^2} = -\mu.PA \text{ (Since the acceleration is towards A)}$$

$$= -\mu(x-y) \qquad (1)$$

The equation of motion of A is $\dfrac{d^2y}{dt^2} = \alpha$ (2)

Subtracting (2) from (1)

We have

$$\frac{d^2x}{dt^2} - \frac{d^2y}{dt^2} = -\mu(x-y) - \alpha = -\mu\left(x - y + \frac{\alpha}{\mu}\right) \qquad (3)$$

Put $x - y + \dfrac{\alpha}{\mu} = z$ (4)

Differentiating (4) $\dfrac{d^2x}{dt^2} - \dfrac{d^2y}{dt^2} = \dfrac{d^2z}{dt^2}$

Hence (3) becomes

$$\frac{d^2z}{dt^2} = -\mu z \qquad (5)$$

Space for Hints

$\therefore$ The displacement z is simple harmonic and the period is $= \dfrac{2\pi}{\sqrt{\mu}}$.

The centre of the S.H.M represented by (5) is clearly the new origin from where z is measured.

Now $z = x - y + \dfrac{\alpha}{\mu} = AP + \dfrac{\alpha}{\mu}$.

If B is a point behind A such that

$$BA = \frac{\alpha}{\mu}.$$

We have $z = AP + BA = BP$.

(ie) z denotes the displacement of P measured from B.

$\therefore$ The motion of P is simple harmonic about B, a moving centre which is always at a distance $\dfrac{\alpha}{\mu}$ behind A.

Example: 5

A particle of mass m is oscillating in a straight line about a centre of force O, towards which when at a distance r, the force is $m.n^2r$ and 'a' is the amplitude of the oscillation. When at a distance $a\sqrt{3}/2$ from O, the particle receives a blow in the direction of motion which generates a velocity na. If this velocity be away from O, show that the new amplitude is a $\sqrt{3}$.

Solution:

Here the constant μ of the S.H.M $= n^2$.

If v is the velocity at a distance x from o,

$v^2 = n^2 (a^2 - x^2)$ since amplitude = a.

When $x = \dfrac{a\sqrt{3}}{2}, v^2 = n^2\left(a^2 - \dfrac{3a^2}{4}\right) = \dfrac{n^2a^2}{4}$

(ie) $v = \pm na/2$

Since the additional velocity now given to the particle is away from O is the direction of its previous motion.

We must take the positive sign for v.

Hence the new total velocity $= \dfrac{na}{2} + na$

$$= \frac{3na}{2}$$

The subsequent motion is again simple harmonic.

If v is the velocity at any distance x from O, is the subsequent motion,

$$v^2 = n^2 (A^2 - x^2) \qquad (1)$$

Where A is the new amplitude.

The new initial condition is that

When $x = \frac{a\sqrt{3}}{2}, V = \frac{3na}{2}$

Substituting in (1)

$$\frac{9n^2a^2}{4} = n^2\left(A^2 - \frac{3a^2}{4}\right)$$

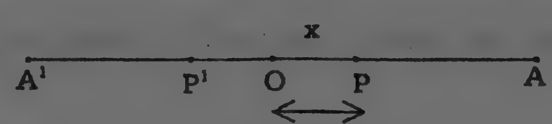
(ie) $A^2 = \frac{9a^2}{4} + \frac{3a^2}{4} = 3a^2$

Or $A = a\sqrt{3}$ Which is the new amplitude

Example: 6

Show that the energy of a system executing S.H.M is proportional to the square of the amplitude and of the frequency.

Solution:



The acceleration of the particle executing S.H.M at a distance x from O = μx.

Hence the force acting on the particle

$= m\mu x$, m being its mass.

If the particle is given a slight displacement dx from P, work done against the force

$= m\mu x.dx$.

Hence total work done is displacing the particle to a distance

$$x = \int_0^x m\mu x\, dx = m\mu\frac{x^2}{2} \qquad (1)$$

This work done is stored as the potential energy of the particle at P.

If v is the velocity at P.

Space for Hints

We know that $v^2 = \mu(a^2 - x^2)$, a being the amplitude

$\therefore$ Kinetic energy at $P = \frac{1}{2}mv^2 = \frac{1}{2}m\mu(a^2 - x^2)$ (2)

Adding (1) and (2),

The total energy at P

$$= \frac{m\mu^2}{2} + \frac{m\mu}{2}(a^2 - x^2) = \frac{m\mu a^2}{2} \quad (3)$$

If n is the frequency.

We know that $n = \frac{1}{\text{Period}} = \frac{1}{\frac{2\pi}{\sqrt{\mu}}} = \frac{\sqrt{\mu}}{2\pi}$

$$\therefore \sqrt{\mu} = 2\pi n \text{ or } \sqrt{\mu} = 4\pi^2 n^2.$$

Putting this values of in (3),

The total energy = ½ m. $4\pi^2 n^2 a^2 = 2\pi^2 ma^2 n^2$ and this is proportional to a^2 and n^2.

Example: 7

Show that in S.H.M of amplitude a and period T the velocity v at a distance x from the centre is given by the relation $v^2 T^2 = 4\pi^2 (a^2 - x^2)$.

Find the new amplitude if the velocity were doubled when the particle is at a distance a/2 from the centre, the period remaining the same.

Solution:

We know that $v^2 = \mu(a^2 - x^2)$

And periodic is $T = \frac{2\pi}{\sqrt{\mu}}$.

$$\therefore \mu = \frac{4\pi^2}{T^2}$$

$$\therefore v^2 = \frac{4\pi^2}{T^2}(a^2 - x^2)$$

Or $v^2 T^2 = 4\pi^2 (a^2 - x^2)$.

The velocity v_1 of the particle at $x = \frac{a}{2}$ is given by

$$v_1^2 = \mu\left(a^2 - \frac{a^2}{4}\right) = 3\mu\frac{a^2}{4}$$

$$\therefore v_1 = (\sqrt{3\mu})\frac{a}{2}$$

Now the initial condition will be that when x = a/2

$$V = 2v_1 = 2\ (\sqrt{3\mu})\frac{a}{2} = (\sqrt{3\mu})a$$

From the equation of S.H.M

We get $\frac{d^2x}{dt^2} = V\frac{dv}{dx} = -\mu x$

$$vdv = -\ \mu x\, dx$$

Integrating on both sides we get $v^2 = -\mu x^2 + A.$ (1)

Using the condition that at $x = \frac{a}{2}, v = 2v_1 = \sqrt{3\mu}a$ (2)

From (1) and (2) we get

$$(\sqrt{3\mu}a)^2 = -\mu\left(\frac{a}{2}\right)^2 + A$$

$$\Rightarrow 3\mu a^2 = \frac{-\mu a^2}{4} + A$$

$$\therefore A = 13\mu\frac{a^2}{4}.$$

$$\therefore \quad v^2 = -\mu x^2 + 13\mu\frac{a^2}{4} = \mu\left(13\frac{a^2}{4} - x^2\right)$$

Amplitude is that value of x where velocity is zero.

$$13\frac{a^2}{4} - x^2 = 0 \text{ (or) } x = \frac{\sqrt{13}}{2}a.$$

Example: 8

If v_1 and v_2 are the velocities of a particle moving is S.H.M at distance x_1 and x_2 from the centre. Show that the time of complete oscillation in $2\pi\left(\sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}\right)$.

Solution:

We know that $v^2 = \mu(a^2 - x^2)$

$$\therefore v_1^2 = \mu(a^2 - x_1^2);\ v_2^2 = \mu(a^2 - x\frac{2}{2})$$

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$$\therefore v_2^2 - v_1^2 = \mu(x_1^2 - x_2^2)$$

$$\therefore \mu = \frac{v_2^2 - v_1^2}{x_1^2 - x_2^2} \qquad (1)$$

Also periodic time = $\dfrac{2\pi}{\sqrt{\mu}} = 2\pi\sqrt{\dfrac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}$ by (1)

Example: 9

The position of a particle moving in a straight line is given x = a cos nt + b sinnt. Prove that it excutes S.H.M of period $\dfrac{2\pi}{n}$ and amplitude $\sqrt{(a^2 + b^2)}$.

Solution:

Differentiating the given relation.

We get

$$v = \frac{dx}{dt} = -an\ \text{Sinnt} + bn\ \text{Cosnt}$$

$$\therefore \frac{d^2x}{dt^2} = -an^2\ \text{Cosnt} - bn^2\ \text{Sinnt} = -n^2x$$

Hence the motion is S.H.M of period

$$\frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{n}$$

$$\therefore \mu = n^2$$

Amplitude is the distance between the centre and the position of instantaneous rest where velocity is zero.

Putting v = 0

We get

$$-\ an\ \text{Sinnt} + bn\ \text{Cosnt} = 0$$

$$\therefore \text{tannt} = \frac{b}{a} \text{ or Sinnt} = \frac{b}{\sqrt{a^2 + b^2}}$$

And Cos nt = $\dfrac{a}{\sqrt{a^2 + b^2}}$

Putting for sinnt and cosnt in

x = a cosnt + b Sinnt

We get Amplitude = $a\dfrac{a}{\sqrt{a^2 + b^2}} + b.\dfrac{b}{\sqrt{a^2 + b^2}}$

Amplitude = $\sqrt{(a^2+b^2}$

Example: 10

A particle moves in a straight line. The velocity v at a distance x from a fixed point in the line is given by $v^2 = \alpha-\beta x^2$. Where α and β are positive constants. Show that the motion is simple harmonic. Find period and amplitude.

Solution:

We have $v^2 = \alpha-\beta x^2$

Differentiating both sides W.r.to t.

We get

$$2v\frac{dv}{dt}=-\beta 2x\frac{dx}{dt}$$

$$\left[\because \ddot{x}=\frac{dv}{dt}\right]$$

(ie) $2v\ddot{x} = -2\beta x v$

(or) $\ddot{x} = -\beta x$

This is of the form $\ddot{x} = -n^2 x$ where $n^2 = \beta$. Hence the motion is simple harmonic of period $2\pi/n = \frac{2\pi}{\sqrt{\beta}}$

Further, the velocity vanishes when $\alpha-\beta x^2=0$

(ie) When $x = \pm\sqrt{\alpha/\beta}$

Hence the motion is confined to a distance $\sqrt{\alpha/\beta}$ on either side of the origin.

Hence the amplitude is $\sqrt{\alpha/\beta}$.

Example: 11

The velocity of a particle moving in a straight line is given by the equation $v = k\sqrt{a^2-x^2}$ where k and a are constants and x is the distance of the particular from a fixed point on the line. Prove that the motion is simple harmonic and find its amplitude.

Solution:

$$v = k\sqrt{a^2-x^2}$$

$$v^2 = k^2 (a^2 - x^2)$$

Differentiating both sides w. r. to x,

We get

$$2v.\frac{dv}{dx} = k^2(-2x) = -2k^2x$$

$$v.\frac{dv}{dx} = -k^2x - k^2x \text{ (ie)} \frac{d^2x}{dt^2} = -k^2x$$

$$[\because v^2 = k^2(a^2 - x^2) v, \frac{dv}{dx} = -k^2x]$$

Above equation shows that acceleration varies as the distance of the particles from a fixed point and is directed towards the fixed point and is directed towards the fixed point.

$\therefore$ The motion is simple harmonic of period $\frac{2\pi}{k}$.

Example: 12

A body moving in a straight line OAB with S.H.M has zero velocity at the points A and B whose distance from O are a and b respectively and has a velocity v when half way between them show that complete periods is $\pi(b-a)/v$.

Solution:

Since the velocities at A and B are zero and as such they are the positions of instantaneous rest and AB = OB – OA = b – a

$\therefore$ Amplitude = ½ AB = $\frac{b-a}{2}$ = A say.

Now in S.H.M $v^2 = \mu(A^2 - x^2)$

At the centre x = 0 and V = v given.

$$\therefore V^2 = \mu A^2 = \mu\left(\frac{b-a}{2}\right)^2 \text{ or } \sqrt{\mu} = \frac{2v}{b-a}$$

$$\therefore T = \frac{2\pi}{\sqrt{\mu}} = 2\pi\frac{(b-a)}{2v} = \pi\frac{(b-a)}{v}$$

Example: 13

The maximum velocity of a particle executing S.H.M is 2ft/sec and its period is $\frac{1}{5}$ sec. Find the amplitude.

Solution:

If $\ddot{x} = -n^2 x$ be the equation of S.H.M and a be the amplitude, then the velocity v at a distance x is given by

$V^2 = n^2 (a^2 - x^2)$

The velocity is a maximum when x = 0 (ie) at the origin) the maximum velocity is na.

Further, the period is $\frac{2\pi}{n}$

$$\text{Here } na = 2 \text{ and } \frac{2\pi}{n} = \frac{1}{5}$$

These equations give $a = (1/5\pi)$ ft.

Example: 14

A particle moving with S.H.M of period πsec has a maximum velocity of 2.4 metres/sec. Find the amplitude and the velocity at a distance of 0.9 metres from the central position.

Solution:

We have $\frac{2\pi}{n} = \pi$

and na = 2.4 and

hence a = 1.2 metres.

Also $v^2 = n^2 (a^2 - x^2)$

$\therefore$ when x = 0.9

We have $v^2 = 2^2 [(1.2)^2 - (0.9)^2]$

$\therefore v = \sqrt{2.52}$ metres/sec

Example: 15

A particle is performing S.H.M of period T about a centre O and it passes through a point P with velocity v in the direction OP. Show that the time elapses before it returns to P is $\frac{T}{\pi}\tan^{-1}(vt/2\pi.op)$

Solution:

Let the equation of motion be

$$\ddot{x} = -n^2 x \text{ where } \frac{2\pi}{n} = T$$

The equation can be written as

Space for Hints

$$(D^2 + n^2)\, x = 0 \text{ where } D = \frac{d}{dt}$$

The solution of this second order linear equation with constant coefficients is given by

$$x = A \cos nt + B \sin nt \qquad (1)$$

Where A and B are arbitrary constants.

Further $\ddot{x} = -An \sin nt + Bn \cos nt \qquad (2)$

If we measure time from the instant when the particle is at P. We have $x = OP$ and $\dot{x} = v$ when $t = 0$.

Equations (1) and (2) with $t = 0$ then give, $OP = A$ and $v = Bn$

$\therefore x = -n\, OP \sin nt + v \cos nt$

Time to reach the extreme position in the direction OP is obtained. When $x = 0$.

$$\therefore \text{ we have } \tan nt = \frac{v}{n.OP}$$

$$t = \frac{1}{n} \tan^{-1}[v / n.OP]$$

$$\text{(ie)} \qquad t = \frac{T}{2\pi} \tan^{-1}[vT / 2\pi.OP]$$

$\therefore$ Time that elapses before the particle returns to $P = 2t$

$$= \frac{T}{2\pi} \tan^{-1}[vT / 2\pi.OP]$$

Example: 16

A mass m lie on a horizontal shelf making vertical simple harmonic oscillations of amplitude 1 foot and period 1 sec. Show that the mass leaves the shelf at a height of $8/\pi^2$ ft above the mean position that it then finally reaches a height of $\left(\frac{\pi^2}{16} + \frac{4}{\pi^2}\right)$ ft above the mean position.

Solution:

At a height x above the mean position O, the acceleration of the shelf (and hence that of the mass m on the shelf) is $n^2 x$ towards O. The forces acting on the mass are its own weight mg vertically downwards and the reaction R of the shelf vertically upwards.

$\therefore$ The equation of motion of the mass is given by $m(n^2 x) = mg - R$

(ie) $R = mg - mn^2x$.

The particle leaves the shelf when R = a

(ie) when $x = \dfrac{g}{n^2}$

But $\dfrac{2\pi}{n} = \text{Period} = 1\,\text{sec}$

$\therefore \quad n = 2\pi$

Hence the particles leaves the shelf at a height $x = \dfrac{g}{n^2} = \dfrac{3^2}{4\pi} = \dfrac{8}{\pi^2}$ feet above the mean position. Further the velocity v of the mass just when it leaves the shelf is given by

$$v^2 = n^2(a^2 - x^2) = 4\pi^2\left(1 - \left(\frac{8}{\pi^2}\right)^2\right)$$

$$= \left(4\pi^2 - \frac{256}{\pi^2}\right)$$

With v as initial velocity the mass travels upwards a further distance y under the retardation g before its velocity vanishes.

$\therefore \quad$ we have $O = v^2 - 2gy$

$$y = \frac{v^2}{2g} = \frac{4\pi^2 - \frac{256}{\pi^2}}{64}$$

(ie) $$= \left(\frac{\pi^2}{16} - \frac{4}{\pi^2}\right)\text{ft}$$

$\therefore$ The total height to which the mass rises above the mean position

$$= \frac{8}{\pi^2} + y = \frac{8}{\pi^2} + \frac{\pi^2}{16} - \frac{4}{\pi^2}$$

$$= \frac{4}{\pi^2} + \frac{\pi^2}{16}$$

Exercise:

1. A mass of 1 gm vibrates through a millimeter on each side of the midpoint of its path 256 times per sec; if the motion be simple harmonic, find the maximum velocity.

2. In a S.H.M if be the acceleration and v the velocity at any time and T is the periodic time, prove that $f^2 T^2 + 4\pi^2 v^2$ is constant.

3. If a body of mass m executing S.H.M. makes n compete oscillations per sec., Show that the difference of its K.E. When at the centre and when at a distance x from the centre is given by $2m\pi^2 n^2 x^2$.

4. A particle moving in S.H.M along a straight line between the points A,A'. P is a point in AA' such that AP:P A' = 1:3. Show that the time from A to P is half the time taken from P to A'.

5. At the end of 3 consecutive seconds, the distances of a point moving with S.H.M from its mean position measured in the same direction are 1,5,5 units. Show that the period of one complete oscillation is $\frac{2\pi}{2}$ secs where $\text{Cos}\,\theta = \frac{3}{5}$.

9.3 Composition of Simple Harmonic Motion

9.3.1 Composition of two Simple Harmonic Motions of the same period and in the same straight line:

Since the period is dependent only on the constant μ. The two separate S.H.M are expressed by the same differential equations.

$$\frac{d^2x}{dt^2} = -\mu x$$

Let x_1 and x_2 be the displacements for the separate motions. Then we can take

$$x_1 = a_1 \text{Cos}(\sqrt{\mu}t + \varepsilon_1) + a_2 \text{Cos}(\sqrt{\mu}t + \varepsilon_2).$$

Let x be the resultant displacement

Then $x = x_1 + x_2$

$$= a_1 \text{Cos}(\sqrt{\mu}t + \varepsilon_1) + a_2 \text{Cos}(\sqrt{\mu}t + \varepsilon_2)$$

Space for

$$= \cos\sqrt{\mu}t(a_1\cos\varepsilon_1 + a_2\cos\varepsilon_2)$$

$$- \sin \sqrt{\mu}t,(a_1\sin\varepsilon_1 + a_2\sin\varepsilon_2)$$

$$= \cos\sqrt{\mu}t.A\cos\varepsilon - \sin\sqrt{\mu}t.A\sin\varepsilon \quad (1)$$

Where $A \cos \varepsilon = a_1\cos\varepsilon_1 + a_2\cos\varepsilon_2$ (2)

And $A \sin \varepsilon = a_1 \sin\varepsilon_1 + a_2 \sin\varepsilon_2$ (3)

We can find the new constants A and $\in$

Squaring (2) and (3) and adding

$$A^2 = a_1^{\,2} + a_2^{\,2} + 2a_1a_2\cos(\varepsilon_1 - \varepsilon_2) \quad (4)$$

Dividing (3) by (2)

$$\tan \varepsilon = \frac{a_1\sin\varepsilon_1 + a_2\sin\varepsilon_2}{a_1\cos\varepsilon_1 + a_2\cos\varepsilon_2} \quad (5)$$

Now (1) becomes $x = A(\cos\sqrt{\mu}t\cos\varepsilon - \sin\sqrt{\mu}t\sin\varepsilon$

$$= A \cos (\sqrt{\mu}t + \varepsilon) \quad (6)$$

The resultant displacement given by (6) also represents a S.H.M of the same period as the individual motions. A new amplitude is the diagonal of the parallelogram whose sides are the original amplitudes a_1 and a_2 inclined to one another at an angle $\varepsilon_1 - \varepsilon_2$ the difference of the epochs.

9.3.2 Composition of two simple Harmonic Motions of the same period in two perpendicular directions:

If a particle possesses two S.H.M in $\perp^r$ directions and of the same period, we can prove that its path is an ellipse. Take the two $\perp^r$ lines as the axes of X and Y. The displacements of the particle due to separate motions can be taken as

$$x = a_1 \cos \sqrt{\mu}t \quad (1)$$

$$y = a_2 \cos (\sqrt{\mu}t + \varepsilon) \quad (2)$$

The path of the particle is obtained by eliminating t between (1) and (2).

From (2) $\Rightarrow$

$$y = a_2 \cos \sqrt{\mu}t.\cos\varepsilon - a_2\sin\sqrt{\mu}t\sin\varepsilon$$

$$= a_2 \cos \varepsilon\frac{x}{a_1} - a_2\sin\varepsilon\sqrt{1 - \frac{x^2}{a_1^{\,2}}}$$

$$\text{(ie)}\ \frac{y}{a_2} - \frac{x\cos\varepsilon}{a_1} = -\text{Sin}\varepsilon\sqrt{1-\frac{x^2}{a_1^{\ 2}}}$$

Squaring

$$\frac{y^2}{a^2{}_2} + \frac{x^2\cos^2\varepsilon}{a_1^{\ 2}} - \frac{2xy\cos\varepsilon}{a_1a_2} = \text{Sin}^2\varepsilon - \frac{x}{a_1^{\ 2}}\text{Sin}^2\varepsilon$$

$$\text{(ie)}\ \frac{x^2}{a_1^{\ 2}} - \frac{2xy}{a_1a_2}\text{Cos}\varepsilon + \frac{y^2}{a^2{}_2} = \text{Sin}^2\varepsilon \qquad (3)$$

This is of the form $ax^2 + 2hxy + by^2 = \lambda$ (4)

Where $a = \frac{1}{a_1^{\ 2}}, h = -\frac{\text{Cos}\varepsilon}{a_1a_2}, b = \frac{1}{a_2^{\ 2}}$

Clearly (4) represents conic with centre at the origin.

Also. $ab - h^2 = \frac{1}{a_1^{\ 2}a_2^{\ 2}} - \frac{\cos^2\varepsilon}{a_1^{\ 2}a_2^{\ 2}} = \frac{\text{Sin}^2\varepsilon}{a_1^{\ 2}a_2^{\ 2}} = +\text{ve}$

Hence (4) represents an ellipse.

If $\varepsilon = 0$, equation (3) gives $\frac{x}{a_1} - \frac{y}{a_2} = 0$ which is also a straight line.

If $\varepsilon = \pi$, (3) gives $\frac{x}{a^1} + \frac{y}{a^2} = 0$ which is also a straight line.

If $\varepsilon = \frac{\pi}{2}$ which is an ellipse whose principal axes are along the axes of x and y.

If $\varepsilon = \frac{\pi}{2}$ and $a_1 = a_2$, the path in the circle $x^2 + y^2 = a_1^{\ 2}$.

Example: 1

Show that the resultant of two S.H.M in the same direction and of equal periodic time, the amplitude of one being twice that of the other and its phase a quarter of a period in advance, is a S.H.M of amplitude $\sqrt{5}$ times that of the first and whose phase is in advance of the first by

$\frac{\tan^{-1}2}{2\pi}$ of a period.

Solution:

Referring to composition of two S.H.M of the same period and in the same straight line.

Let the separate displacements be

$x_1 = a_1 \cos(\sqrt{\mu}t + \varepsilon_1)$ and (1)

$x_2 = a_2 \cos(\sqrt{\mu}t + \varepsilon_2)$ (2)

Here $a_2 = 2a_1$ and $\frac{\varepsilon_2 - \varepsilon_1}{\sqrt{\mu}} =$ Phase difference

$$= \frac{1}{4} x \frac{2\pi}{\sqrt{\mu}}$$

$\therefore \varepsilon_2 - \varepsilon_1 = \frac{\pi}{2}$ or $\varepsilon_2 = \frac{\pi}{2} + \varepsilon_1$

We know that the resultant displacement is

$x = A \cos(\sqrt{\mu}t + \varepsilon)$ (3)

Where $A^2 = a_1{}^2 + a_2{}^2 + 2a_1 + a_2 \cos(\varepsilon_1 - \varepsilon_2)$

$= a_1{}^2 + 4a_1{}^2 + 4a_1{}^2 \text{Cos}(-90^0) = 5a_1{}^2$

$\therefore$ Amplitude of the resultant motion = $A = a_1\sqrt{5}$.

Also $\tan = \varepsilon = \frac{a_1 \text{Sin}\varepsilon_1 + a_2 \text{Sin}\varepsilon_2}{a_1 \cos\varepsilon_1}$

$$= \frac{a_1 \text{Sin}\varepsilon_1 + 2a_1 \text{Sin}(90^0 + \varepsilon_1)}{a_1 \cos\varepsilon_1 + 2a_1 \cos(90^0 + \varepsilon_1)}$$

(ie) $\frac{\sin\varepsilon}{\cos\varepsilon} = \frac{\sin\varepsilon_1 + 2\text{Cos}\varepsilon_1}{\cos\varepsilon_1 - 2\text{Sin}\varepsilon_1}$

$\text{Sin } \varepsilon \cos\varepsilon_1 - 2\text{Sin}\varepsilon \sin\varepsilon_1 = \sin\varepsilon_1 \cos\varepsilon + 2\text{Cos}\varepsilon_1 . \cos\varepsilon$

Or $\sin(\varepsilon - \varepsilon_1) = 2\cos(\varepsilon - \varepsilon_1)$

(ie) $\tan \varepsilon - \varepsilon_1 = 2$

(or) $\varepsilon - \varepsilon^1 = \tan^{-1} 2$

$$\therefore \frac{\varepsilon - \varepsilon_1}{\sqrt{\mu}} = \frac{\tan^{-1} 2}{2\pi}\left(\frac{2\pi}{\sqrt{\mu}}\right)$$

$$= \frac{\tan^{-1} 2}{2\pi} \text{of a period.}$$

This is the phase difference of the resultant simple harmonic motion.

Space for Hints

Exercise:

1) Two simple harmonic motions in the same straight line of equal periods and differing in phase by $\frac{\pi}{2}$ are impressed simultaneously on a particle. If the amplitudes are 4 and 6. find the amplitude and phase of the resulting motion.

2) A particle possesses two S.H.M of the same period with amplitudes a and b and phase difference $\frac{\pi}{2}$ is two $\perp^r$ directions. Show that the particle traces an ellipse whose semi major and minor axes are a and b.

UNIT – 10

CENTRAL ORBITS

10.1 Introduction:

In the previous chapters, we have considered some particular case of motion of a particle in two dimensions. To fix the position of a particle in a plane, we require two coordinates and to study the motion of the particle, we require its component velocities and accelerations in two mutually $\perp^r$ directions. We had previously used cartesian coordinates. In this chapter we shall use polar coordinates.

10.2 Velocity and Acceleration in polar coordinates:

Let P be the position of a moving particle at time t. Taking O as the pole and OX as the initial line, let the polar coordinates of P be (r, θ). $\overline{OP} = r$ is the position vector of P. Hence the velocity of P = $\frac{d}{dr}(r)$. Since r has modulus r and amplitude $\theta, \frac{d}{dt}(r)$ will have components $\dot{r}$ along OP and $r\dot{\theta}$ to OP.

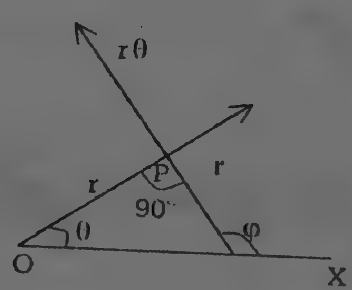


Fig. (9.1)

Hence the velocity vector v at P has components $\dot{r}$ along OP in the direction in which r increases and $r\dot{\theta} \perp^r$ to OP.

In the direction in which θ increases. These are respectively called the radial and transverse components of v.

The acceleration vector at P is the derivative of the velocity vector v. The radial component of v is a vector which modulus r and amplitude θ. Hence the derivative of $\dot{r}$ will have components.

i) $\frac{d}{dt}(\dot{r}) = \ddot{r}$

along OP in the direction in which r increases and

Space for Hints

ii) $\dot{r}\dfrac{d}{dt}(\theta) = \dot{r}\dot{\theta}$

$\perp^r$ to OP in the direction in which θ increases. This is shown in fig (9.2)

The transverse component of v is a vector with modulus $r\dot{\theta}$ and amplitude

$\phi = \dfrac{\pi}{2} + \theta$.

Hence the derivative of $r\dot{\theta}$ with have components.

(i) $\dfrac{d}{dt}(r\dot{\theta}) = r\ddot{\theta} + \dot{\theta}\dot{r}$ along the line of $r\dot{\theta}$

and (ii) $r\dot{\theta}\dfrac{d}{dt}\left(\dfrac{\pi}{2} + \theta\right) = r\dot{\theta}^2$

in the direction $\perp^r$ to the line of $r\dot{\theta}$.

(ie) in the direction PO.

(This component is towards O, as it is in the direction in which ϕ increases)

This is shown in fig. (9.3)

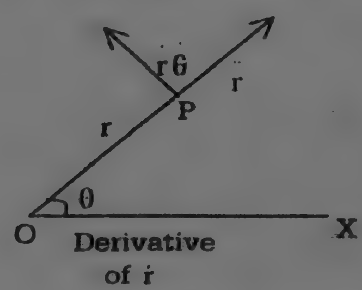


Fig.(9.2)

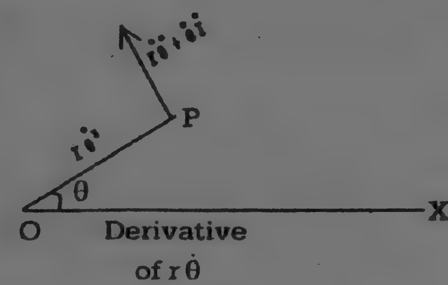


Fig.(9.3)

Hence the totals of the components of acceleration are $\ddot{r} - r\dot{\theta}^2$ in direction OP and $r\ddot{\theta} + 2\dot{r}\dot{\theta}$ in the $\perp^r$ direction.

$$\text{Now, } \frac{1}{r}.\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{r}(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta})$$

$$= r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$\therefore$ Acceleration $\perp^r$ to OP is also $= \dfrac{1}{r}\dfrac{d}{dt}(r^2\dot{\theta})$.

The above results are collected in a table of reference.

		Magnitude	Direction	Sense
1.	Radial component of velocity	$\dot{r}$	Along the radius vector	In the direction in which r increases.
2.	Transverse component of velocity	$r\dot{\theta}$	Perpendicular to the radius vector.	In the direction in which θ increases.
3.	Radial component of acceleration	$\ddot{r} - r\dot{\theta}^2$	Along the radius vector	In the direction in which r increases
4.	Transverse component Of acceleration	$\frac{1}{r}.\frac{d}{dt}(r^2\dot{\theta})$	Perpendicular to the radius vector.	In the direction in which θ increases.

Theorem: 10.2.1

The velocities of a particle along and $\perp^r$ to the radius vector from a fixed point are a and b. Find the acceleration components along $\perp^r$ to the radius vector and the equation of the path.

Solution:

We are given that $\dot{r} = a$ and $r\dot{\theta} = b$.

$$\therefore \frac{\dot{r}}{r\dot{\theta}} = \frac{a}{b}$$

$$\text{(ie)}\ \frac{dr}{rd\theta} = \frac{a}{b} \qquad [\therefore \dot{r} = \frac{dr}{dt}\ \&\ \dot{\theta} = \frac{d\theta}{dt}]$$

$$\text{(ie)}\ \frac{dr}{r} = \frac{a}{b}d\theta$$

On integration, we get

$$\log r = \frac{a}{b}\theta + \log K$$

$$\text{(ie)}\ \log\left(\frac{r}{k}\right) = \frac{a}{b}\theta$$

(ie) $r = ke^{a\theta/b}$ which is an equiangular spiral.

Further, since $r = a$

We have $\dot{r} = 0$

$\therefore$ the radial acceleration $= \ddot{r} - r\dot{\theta}^2$

$$= 0 - \frac{1}{r}(r\dot{\theta})^2$$

$$= -\frac{b^2}{r}.$$

And the transverse acceleration

$$= \frac{1}{r}.\frac{d}{dt}(r^2\dot{\theta})$$

Transverse acceleration $= \frac{1}{r}.\frac{d}{dt}(r.r\dot{\theta})$

$$= \frac{1}{r}\frac{d}{dt}(r.b) = \frac{1}{r}b\dot{r}$$

$$= \frac{1}{r}ba = \frac{ab}{r}. \qquad (\therefore \dot{r} = a)$$

Corollary:

The magnitude of the resultant velocity of P

$$= \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$$

And the magnitude of the resultant acceleration

$$= \sqrt{\left(\ddot{r} - r\dot{\theta}^2\right)^2 + \left\{\frac{1}{r}.\frac{d}{dt}(r^2\dot{\theta})\right\}^2}$$

10.2.2 Equations of Motion in polar Coordinates:

If R and S are the components of the external force acting on a particle of mass m is the radial and transverse directions, we have the equation

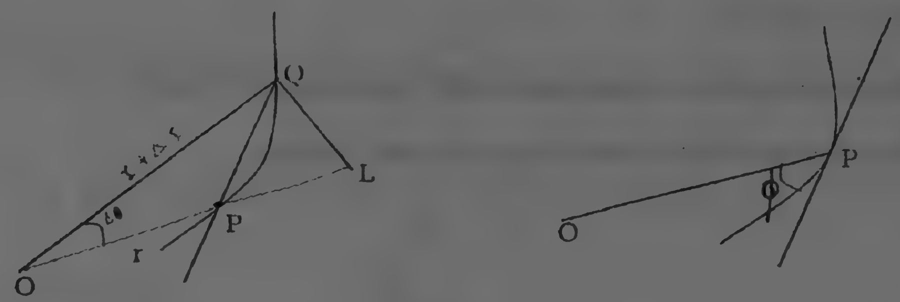
$$R = m\ (\ddot{r} - r\dot{\theta})^2 \qquad (1)$$

And $$S = m.\frac{1}{r}.\frac{d}{dt}\left(r^2\dot{\theta}\right) \qquad (2)$$

If R and S are known functions of the coordinates r, θ and the time t, the differential equations (1) and (2) can be solved to find r and θ as functions of t and by eliminating t, the polar equation to the path is got.

Note on the equiangular spiral:

Some equations in this chapter will relates to the curve called the equiangular spiral. This curve has the important property that the tangent at any point P on, it makes a constant angle with the radius vector OP.



Let OP (=r) and OQ (=r + Δ r) be two consecutive radii vectors such that the included angle POQ = $\Delta\,\theta$.

Draw QL $\perp^r$ to OP.

Then OL = (r + Δ r) Cos $\Delta\,\theta$ = r + Δ r approximately.

Hence PL = OL – OP = Δ r

And LQ = (r + Δ r) Sin $\Delta\,\theta$ = (r + Δ r) $\Delta\,\theta$ nearly

= r $\Delta\,\theta$ to the first order of smallness

Hence $\tan\angle QPL = \dfrac{QL}{PL} = r\dfrac{\Delta\theta}{\Delta r}$.

In the limit as Δ r and $\Delta\,\theta$ both $\to$ O the point Q tends to coincide with P. The chord QP becomes in the limiting position, the tangent at P. Let ϕ be the angle made by the tangent at P with OP.

Then $\phi = \underset{Q\to P}{Lt}\angle QPL$

Hence $\tan\phi = \underset{\Delta r\to 0}{Lt}\tan\angle QPL$

$= \underset{\Delta r\to 0}{Lt}\, r\dfrac{\Delta\theta}{\Delta r}$

$= r\dfrac{d\theta}{dr}$

This formula is an important one in dealing with curves in polar coordinates and it gives the angle between the radius vector and the tangent. Now for the equiangular spiral, at any point P on it the angle ϕ is constant.

Let $\phi = \alpha$. Then $\tan\phi = \tan\,\alpha$.

$\therefore$

(ie) $r\frac{d\theta}{dr} = \tan\alpha.$

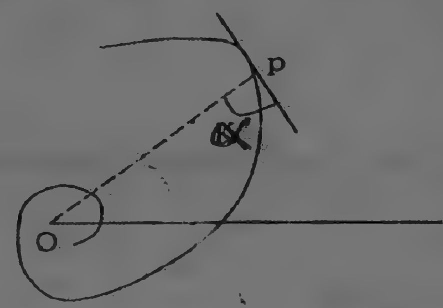
(or) $\frac{dr}{r} = \text{Cot}\,\alpha.d\theta$

Integrating. $\log r = \theta\ \text{Cot}\ \alpha + \text{constant}$

(ie) $r = ae^{\theta\cot\alpha}$

This is the polar equation to the equiangular spiral

This shape of the curve is as shown below.



Worked Examples

Example: 1

The velocities of a particle along and $\perp^r$ to a radius vector from a fixed origin are λr^2 and $\mu\theta$ where μ and λ are constants. Show that the equation to the path of the particle is $\frac{\lambda}{\theta}+C=\frac{\mu}{2r^2}$ where C is a constant.

Show also that the accelerations along and perpendicular to the radius vector are $2\ \lambda^2 r^3 - \frac{\mu^2\theta^4}{r}$ and $\mu\left(\lambda r\theta^2 + \frac{2\mu\theta^3}{r}\right)$.

Solution:

$$\text{Radial velocity} = \frac{dr}{dt} = \lambda r^2 \qquad (1)$$

$$\text{Transverse velocity} = r\frac{d\theta}{dt} = \mu\theta^2 \qquad (2)$$

Dividing (2) by (1)

We have

$$r\frac{d\theta}{dr} = \frac{\mu\theta^2}{\lambda r^2} \qquad \text{(ie)}\ \lambda\frac{d\theta}{\theta^2} = \frac{\mu}{r^3}dr$$

Integrating $-\frac{\lambda}{\theta} = -\frac{\mu}{2r^2} + C$

(ie) $\dfrac{\mu}{2r^2}=\dfrac{\lambda}{\theta}+C$ (3)

(3) is the equation to the path.

Differenting (1), $\dfrac{d^2r}{dt^2}=\lambda\ 2r\dfrac{dr}{dt}=2\ \lambda^2r^3(\because \text{by}(1))$

Radial acceleration = $\ddot{r}-r\dot{\theta}^2=\dfrac{d^2r}{dt^2}-r\left(\dfrac{d\theta}{dt}\right)^2$

$$=2\lambda^2r^3-r\left(\frac{\mu\theta}{r}\right)^2=2\lambda^2r^3-\frac{\mu^2\theta^4}{r} \text{ using (2)}$$

Transverse acceleration

$$=\frac{1}{r}.\frac{d}{dt}(r^2\dot{\theta})=\frac{1}{r}.\frac{d}{dt}\left(r^2\frac{\mu\theta^2}{r}\right)$$

$$=\frac{1}{r}.\frac{d}{dt}(\mu r\theta^2)=\frac{\mu}{r}\left[r2\theta\frac{d\theta}{dt}+\theta^2\frac{dr}{dt}\right]$$

$$=\frac{\mu}{r}\left[2r.\theta\frac{\mu\theta^2}{r}+\theta^2.\lambda r^2\right]=\mu\left[\frac{2\mu\theta^3}{r}+\lambda r\theta^2\right]$$

Example: 2

Show that the path of a point P which possesses two constant velocities u and v, the first of which is in a fixed direction and the second of which is perpendicular to the radius OP drawn from a fixed point O, is a conic whose focus is O and whose eccentricity is $\dfrac{u}{v}$.

Solution:

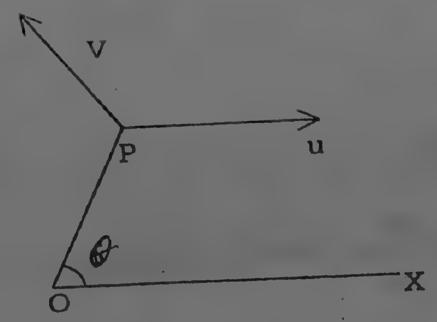
Take O as the pole and the line OX parallel to the given direction as the initial line. P has two velocities u parallel to OX and v perpendicular to OP.

Resolving the velocities along and perpendicular to OP.

We have $\dfrac{dr}{dt}=u\cos\theta$ (1)

$r\dfrac{d\theta}{dr}=v-u\sin\theta$ (2)

To get the equation to the path, we have to eliminate t.



Dividing (2) by (1)

We have

$$r\frac{d\theta}{dr}=\frac{v-u\,Sin\,\theta}{u\,Cos\,\theta}$$

(ie) $\dfrac{u\,Cos\,\theta}{v-u\,Sin\,\theta}d\theta = \dfrac{dr}{r}$ or $\dfrac{d\,(u\sin\theta)}{v-u\sin\theta}=\dfrac{dr}{r}$

Integrating,

- log (v – u sin θ) + log A = log r, where log A is the constant of integration

(ie) $\log r+\log(v-u\,Sin\;\theta)=\log A$

(ie) $r\,(v-u\,Sin\;\theta) = A$ or $\dfrac{A}{r}=v-u\,Sin\,\theta$

(ie) $\left(\dfrac{A}{v}\right)\dfrac{1}{r}=1-\dfrac{u}{v}Sin\theta=1+\dfrac{u}{v}Cos(90^0+\theta)$ (1)

This is the form $\dfrac{\ell}{r}=1+e\,\cos(\theta+\alpha)$ (2)

Comparing (1) and (2)

We have $\ell = \dfrac{A}{v}.e=\dfrac{u}{v}$ and $\alpha=90^0$

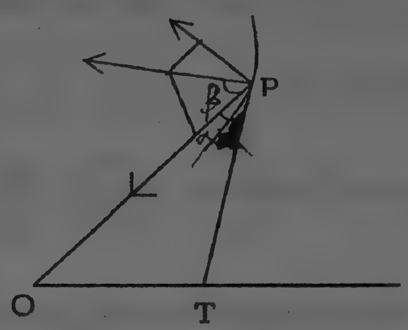
We know from Analytical Geometry that (2) is the polar equation to a conic whose focus is at the pole. Semi – latus rectum is ℓ, eccentricity is e and whose major axis makes an angle α, with the initial line. Hence (1) is a conic whose focus is at O, semi – latus rectum is $\dfrac{A}{v}$, eccentricity is $\dfrac{u}{v}$ and whose major axis is perpendicular to the initial line.

Example: 3

A point P describes a curve with constant velocity and its angular velocity about a given fixed point O varies inversely as the distance from O; show that the curve is an equiangular spiral whose pole is O and that the acceleration of the point is along the normal at P and varies inversely as OP.

Solution:

Taking O as the pole, Let P be (r, θ).



Resultant velocity of P

$$= \sqrt{\dot{r}^2 + r^2\dot{\theta}^2} = \text{constant (given)}$$

If this velocity = k

We have

$$\dot{r}^2 + r^2\dot{\theta}^2 = k^2 \qquad (1)$$

Also angular velocity about O

$$\dot{\theta} = \frac{\lambda}{r} \text{(given)} \qquad (2)$$

From (1) and (2)

$$\dot{r}^2 + r^2\frac{\lambda^2}{r^2} = k^2$$

(ie) $\dot{r}^2 = k^2 - \lambda^2$ or $\dot{r} = \sqrt{k^2 - \lambda^2}$ (3)

Eliminating from (2) and (3)

$$\frac{dr}{d\theta} = \frac{\left(\frac{dr}{dt}\right)}{\left(\frac{d\theta}{dt}\right)} = \frac{\dot{r}}{\dot{\theta}} = \frac{\sqrt{k^2 - \lambda^2}}{\lambda} r$$

(ie) $\dfrac{dr}{r} = \dfrac{\sqrt{k^2 - \lambda^2}}{\lambda} d\theta$

Space for Hints

Integrating $\log r = \frac{\sqrt{k^2-\lambda^2}}{\lambda}\theta + B$

(or) $r = e^{\frac{\sqrt{k^2-\lambda^2}}{\lambda}\theta + B} = e^B . e^{\frac{\sqrt{k^2-\lambda^2}}{\lambda}\theta}$

Putting $e^B = a$ and $\frac{\sqrt{k^2-\lambda^2}}{\lambda} = \cot\alpha$

The above becomes $r = ae^{\theta\cot\alpha}$ (4)

Hence the path is an equiangular spiral, whose pole is O.

Differentiating (3) . $\ddot{r} = 0$

Radial acceleration $= \ddot{r} - r\dot{\theta}^2 = -r\frac{\lambda^2}{\dot{r}^2} = -\frac{\lambda^2}{r}$

The negative sign shows that the radial acceleration at P is along PO.

Transverse acceleration

$$= \frac{1}{r}.\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{r}.\frac{d}{dt}\left(r^2\frac{\lambda}{r}\right) = \frac{1}{r}.\frac{d}{dt}(r\lambda)$$

$$= \frac{\lambda}{r}\dot{r} = \frac{\lambda}{r}\sqrt{k^2-r^2} \text{ from (3)}$$

Resultant acceleration of P

$$= \sqrt{\left(\frac{-\lambda^2}{r}\right)^2 + \left[\frac{\lambda}{r}\sqrt{k^2-\lambda^2}\right]^2}$$

$$= \sqrt{\frac{\lambda^4}{r^2} + \frac{\lambda^2}{r^2}(k^2-\lambda^2)} = \sqrt{\frac{\lambda^2 k^2}{r^2}} = \frac{\lambda k}{r}.$$

Thus the resultant acceleration varies inversely as r.

(ie) as OP. Let this acceleration be along PN making an angle β with PO.

$$\tan\beta = \frac{\text{Component } \perp^r \text{ to PO}}{\text{Component along PO}}$$

$$= \frac{\frac{\lambda}{r}\sqrt{k^2-\lambda^2}}{\lambda^2} = \frac{\sqrt{k^2-\lambda^2}}{\lambda}$$

But $\cot\alpha = \frac{\sqrt{k^2-\lambda^2}}{\lambda}$ from equation (4)

Hence $\tan\beta = \cot\alpha = \tan(90^0 - \alpha)$

(ie) $\beta = 90^0 - \alpha$ or $\beta + \alpha = 90^0$

Hence $\angle NPT = 90^0$ where PT is the tangent at P.

Hence PN is normal at P.

Example: 4

A smooth straight thin tube revolves with uniform angular velocity ω in a vertical plane about one extremity which is fixed. If at zero time the tube be horizontal, and a particle inside it be at a distance 'a' from the fixed end, and be moving with velocity v along the tube show that the distance at time t is.

$$a \operatorname{Cos h} \omega t + \left(\frac{v}{\omega} - \frac{g}{2\omega^2}\right)\operatorname{Sin}\omega t + \frac{g}{2\omega^2}\sin\omega t .$$

Solution:

Let at time t, P be the position of the particle of mass m on the tube OB.

The forces acting at P are

i) its weight mg vertically downwards and

ii) normal reaction $R \perp^r$ to OB.

Let P be (r, θ).

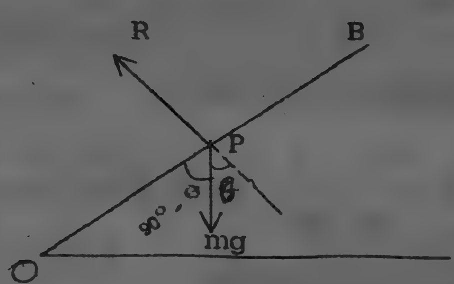
$$\text{Angular velocity} = \dot{\theta} = \frac{d\theta}{dt} = \omega \text{ (given)}$$

Integrating $\theta = \omega t + A$

Initially when $t = o$, $\theta = 0$

$\therefore A = 0$

$$\therefore \theta = \omega t \qquad (1)$$



Resolving along the radius vector OB.

$$m(\ddot{r} - r\dot{\theta}^2) = -mg \operatorname{Cos}(90^0 - \theta) = -mg\sin\theta$$

$$\therefore \ddot{r} - r\omega^2 = -g\operatorname{Sin}\theta = -g\operatorname{Sin}\omega t \qquad \text{(using (1))}$$

$$\text{(ie) } (D^2 - \omega^2)\, r = -g \operatorname{Sin} \omega t \qquad (2)$$

Space for Hints

Where $D = \frac{d}{dt}$

The complementary function Y is found such that $(D^2 - \omega^2) Y = 0$

The solution of this differential equation is $Y = Ae^{\omega t} + Be^{-\omega t}$ (3)

Where A and B are constants.

The particular integral u of the equation (2) is given by

$(D^2 - \omega^2) u = -g \operatorname{Sin} \omega t$

$$\therefore u = -\frac{g}{D^2 - \omega^2} \operatorname{Sin} \omega t = -\frac{g}{-\omega^2 - \omega^2} \operatorname{Sin} \omega t$$

$$u = \frac{g}{2\omega^2} \operatorname{Sin} \omega t \tag{4}$$

Hence the general solution of (2) is

$$r = Y + u = Ae^{\omega t} + Be^{-\omega t} + \frac{g}{2\omega^2} \operatorname{Sin} \omega t \tag{5}$$

The initial conditions are: when $t = 0$

$r = a$ and $\dot{r} = V$

Hence (5) gives $A + B = a$ (6)

Differentiation (5),

$$\dot{r} = A\omega e^{\omega t} - B\omega e^{-\omega t} + \frac{g}{2\omega} \cos \omega t \tag{7}$$

Putting $t = 0$ and $\dot{r} = V$ in (7)

We have

$$A\omega - B\omega + \frac{g}{2\omega} = V \text{ (or) } A - B = \frac{V}{\omega} - \frac{g}{2\omega^2} \tag{8}$$

Solving (6) and (8)

$$A = \frac{1}{2}\left(a + \frac{V}{\omega} - \frac{g}{2\omega^2}\right) \text{ and } B = \frac{1}{2}\left(a - \frac{V}{\omega} + \frac{g}{2\omega^2}\right)$$

Substituting these values in (5)

$$r = \frac{1}{2}\left(a + \frac{V}{\omega} - \frac{g}{2\omega^2}\right)e^{\omega t} + \frac{1}{2}\left(a - \frac{V}{\omega} + \frac{g}{2\omega^2}\right)e^{-\omega t} + \frac{g}{2\omega^2} \operatorname{Sin} \omega t$$

$$= \frac{a(e^{\omega t} + e^{-\omega t})}{2} + \left(\frac{V}{\omega} - \frac{g}{2\omega^2}\right)\left(\frac{e^{\omega t} - e^{-\omega t}}{2}\right) + \frac{g}{2\omega^2} \operatorname{Sin} \omega t$$

$$r = a\,\mathrm{Cos\,h}\,\omega t + \left(\frac{V}{\omega} - \frac{g}{2\omega^2}\right)\mathrm{Sin\,h}\,\omega t + \frac{g}{2\omega^2}\mathrm{Sin}\,\omega t.$$

Example: 5

A particle moves with uniform speed v along a cardioid r = a (1 + Cos θ) Show that (i) its angular velocity about the pole is {v sec (θ/2)/2a} (ii) the radial acceleration is constant and equal to $\frac{-3v^2}{4a}$ and (iii) the magnitude of the resultant acceleration is $\frac{3vw}{2}$.

Solution:

Since r = a (1 + Cos θ)

We have $\dot{r} = a\,\mathrm{Sin}\,\theta\ \dot{\theta}$

∴ The radial velocity $\dot{r} = -a\mathrm{Sin}\theta\,\dot{\theta}$

The transverse velocity $= r\dot{\theta}$

$$= a\,(1 + \mathrm{Cos}\ \theta)\dot{\theta} \qquad (1)$$

We are given that

$$\dot{r}^2 + r^2\dot{\theta}^2 = v^2$$

(ie) $a^2\,\mathrm{Sin}^2\,\theta\dot{\theta}^2 + a^2(1+\mathrm{Cos}\theta)^2\dot{\theta}^2 = v^2$

(ie) $\dot{\theta}^2\,a^2\left[\mathrm{Sin}^2\theta + (1+\mathrm{Cos}\theta)^2\right] = v^2$

or $\dot{\theta}^2\,.2a^2[1+\mathrm{Cos}\theta] = v^2$

or $\dot{\theta}^2\,.4a^2\,\mathrm{Cos}^2(\theta/2) = v^2$

or $\dot{\theta}.2a\mathrm{Cos}^2(\theta/2) = v$

(ie) $\dot{\theta} = \frac{v}{2a\,\mathrm{Cos}(\theta/2)} = \frac{v\sec(\theta/2)}{2a}$

∴ The angular velocity about the pole is $[v\sec(\theta/2)/2a]$.

Substituting for $\dot{\theta}$ is (1)

We get

Space for Hints

$$\dot{r} = -a\,\text{Sin}\,\theta.\frac{v\,\text{Sec}(\theta/2)}{2a}$$

$$= \text{-a.2 Sin}\frac{\theta}{2}\text{Cos}\frac{\theta}{2}.\frac{v\,\text{Sec}(\theta/2)}{2a}$$

$$\dot{r} = -v\,\text{Sin}(\theta/2)$$

$$\ddot{r} = -v\,\text{Cos}(\theta/2).1/2\dot{\theta}$$

$$\ddot{r} = -v\,\text{Cos}(\theta/2).1/2.\frac{v\,\text{Sec}(\theta/2)}{2a}$$

$$\ddot{r} = -\frac{v^2}{4a}.$$

$\therefore$ Theradial acceleratıon

$$= \ddot{r} - r\dot{\theta}^2$$

$$= -\frac{v^2}{4a} - a(1+\text{Cos}\,\theta)\dot{\theta}^2$$

$$= -\frac{v^2}{4a} - 2a\,\text{Cos}^2\left(\frac{\theta}{2}\right)\left\{\frac{v\,\text{Sec}(\theta/2)}{2a}\right\}^2$$

$$= -\frac{v^2}{4a} - \frac{v^2}{2a} = -\frac{3v^2}{4a}$$

Which is a constant.

Transverse component acceleration

$$= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{a(1+\text{Cos}\theta)}\frac{d}{dt}\left[a^2(1+\text{Cos}\theta)^2\frac{v}{2a\text{Cos}\theta/2}\right]$$

$$= \frac{va^2}{a2\text{Cos}^2\theta/2}\frac{d}{dt}\left(4\text{Cos}^4\frac{\theta}{2}.\frac{1}{2a\text{Cos}\frac{\theta}{2}}\right)$$

$$= \frac{v}{\cos^2\theta/2}\frac{d}{dt}(\text{Cos}^3\theta/2) = -\frac{v}{\text{Cos}^2\theta/2}3\text{Cos}^2\frac{\theta}{2}\text{Sin}\frac{\theta}{2}.\frac{1}{2}\dot{\theta}$$

$$= \frac{-3v}{2}\text{Sin}\theta/2\frac{v}{2a\text{Cos}\theta/2} = \frac{-3v^2}{4a}\tan\theta/2.$$

Magnitude of acceleration = $\sqrt{(\text{rad.acc})^2 + (\text{trans.acc})^2}$

Space for Hints

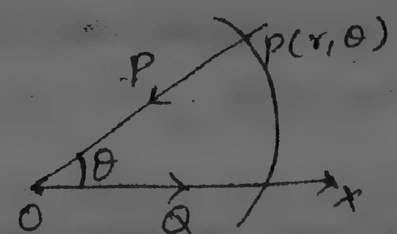
$$=\sqrt{\left(\frac{-3v^2}{4a}\right)^2+\left(\frac{-3v^2}{4a}\tan\theta/2\right)}$$

$$=\frac{3v^2}{4a}\sec\theta/2=\frac{3v}{2}\left(\frac{v}{2a}\sec\theta/2\right)$$

$$=\frac{3vw}{2}.\ [\because \text{by proof of (i)}]$$

Example: 6

A particle of mass m moves along the curve $r = a(1+\cos\theta)$ under the action of a force P towards the pole and a force Q in the positive direction of the initial line. If the angular velocity of the particle about the pole be constant and is equal to ω, find the values of P and Q and show that the K.E of the particle at any point of the path is $a\left(\frac{2P+3Q}{8}\right)$



Solution:

Given that $r = a(1+\cos\theta)$ and $\dot\theta = \omega$

$\therefore$ we have $\dot r = -a\sin\theta.\dot\theta = -a\omega\sin\theta$

and $\ddot r = -a\omega\cos\theta.\dot\theta = -a\omega^2.\cos\theta$

The equations of motion of the particle are

$$m(\ddot r - r\dot\theta^2) = -P + Q\cos\theta \qquad (1)$$

$$\text{And } m.\frac{1}{r}\frac{d}{dt}\left(r^2\dot\theta\right) = -Q\sin\theta \qquad (2)$$

Given that $r = a(1+\cos\theta)$

$$\dot r = -a\sin\theta\dot\theta = -aw\sin\theta$$

$$\ddot r = -a\omega^2\cos\theta$$

$(1) \Rightarrow -P + Q.\cos\theta = -m\omega^2[a\cos\theta + r]$

$$= -m\omega^2[a\cos\theta + a(1+\cos\theta)]$$

Space for Hints

$$= -m\omega^2 a[2\cos\theta + 1] \qquad (3)$$

$$(2) \Rightarrow \; -Q\sin\theta = -2m\omega^2 a\sin\theta \qquad (4)$$

Eliminating Q from (3) and (4) ; (3) x Sin θ + (4) Cos θ

We get

$$-P\sin\theta = -m\omega^2 a[2\sin\theta\cos\theta + \sin\theta] - 2m\omega^2 a\sin\theta\cos\theta$$

$$= -m\omega^2 a\sin\theta\,[4\cos\theta + 1]$$

$$\text{(ie)} \quad P = -m\omega^2 a\,[1 + 4\cos\theta]$$

$$= m\omega^2 a\left[1 + 4.\frac{r-a}{a}\right] \text{ Since } r = a(1+\cos\theta)$$

$$P = m\omega^2 a\,[4r - 3a].$$

Substituting the value of P in (3)

We get

$$Q\cos\theta = P - m\omega^2 a\,(2\cos\theta + 1)$$

$$= m\omega^2 a\,(1 + 4\cos\theta) - m\omega^2 a\,(2\cos\theta + 1)$$

$$Q\cos\theta = 2m\omega^2 a.\cos\theta$$

$$\text{(ie) } Q = 2m\omega^2 a$$

Thus the values of P and Q are determined

Further,

$$\frac{2P+3Q}{8} = \frac{2m\omega^2[4r-3a] + 6m\omega^2 a}{8}$$

$$= m\omega^2 r \qquad (5)$$

The velocity components of the particle along the radial and transverse directions are $\dot{r}$ and $r\dot{\theta}$

$$\therefore \text{ The K.E} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$= \frac{1}{2}m[(a\omega\sin\theta)^2 + a(1+\cos\theta)^2\omega^2]$$

$$= \frac{1}{2}m\omega^2 a^2[\sin^2\theta + (1+\cos\theta)^2]$$

$$= \frac{1}{2}m\omega^2 a^2[2 + 2\cos\theta)^2]$$

$$= \frac{1}{2}m\omega^2 a^2[1 + \cos\theta]$$

Space for Hints

$$K.E = m\omega^2 ar$$

Using (5) in this we see that

$$K.E = a.\left(\frac{2P+3Q}{8}\right)$$

Example: 7

A point P describes an equiangular spiral with constant angular velocity about O. Show that the acceleration varies as OP and has a direction making with the tangent at P the same constant angle which OP makes.

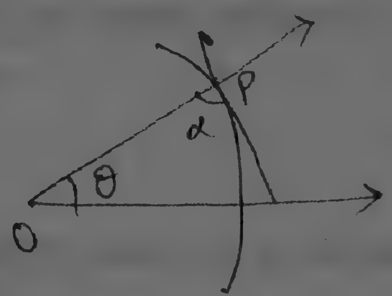
Solution:

Let $r = ae^{\theta\cot\alpha}$ .... (1) be the equation of the equiangular spiral.

So that a is the constant angle which the radius vector OP makes with the tangent at P.

We have

$$\dot{r} = ae^{\theta\cot\alpha}\cot\alpha = r\omega\cot\alpha \quad \left(\therefore \text{by}(1)\,\dot{\theta} = \omega\right) \qquad (2)$$



Where ω is the constant angular velocity about O.

Differentiating once again.

We get $\ddot{r} = \dot{r}\,\omega\cot\alpha = r\omega^2\alpha$ ($\therefore$by(2)

$\therefore$ The radial acceleration of the particle

$$= \ddot{r} - r\dot{\theta}^2$$

$$= r\omega^2\cot^2\alpha - r\omega^2$$

$$= r\omega^2(\cot^2\alpha - 1)$$

The transverse acceleration $= \dfrac{1}{r}\dfrac{d}{dt}(r^2\dot{\theta})$

$$= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) \qquad (\therefore \dot{\theta} = w)$$

Space for Hints

$$=\frac{1}{2}\omega 2r\dot{r}=2\omega r=(r\omega \text{Cot}\,\alpha)=2rw^2\text{Cot}\,\alpha$$

If ϕ be the angle which the resultant acceleration making with the radius vector OP. Then tan ϕ = (Transverse acceleration)/ (Radial acceleration).

$$\therefore \tan\phi=\frac{2\omega^2 r\,\text{Cot}\,\alpha}{r\omega^2(\text{Cot}^2\alpha-1)}=\frac{2\,\text{Cot}\,\alpha}{\text{Cot}^2\alpha-1}=\frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$=\tan 2\alpha$$

$$\therefore \phi \quad =2\alpha$$

(ie) The direction of the acceleration makes with the tangent at P an angle $=(2\alpha - n\alpha)=\alpha$.

(ie) The same constant angle α which OP makes with the tangent at P.

Example: 8

The Velocities of a particle along and perpendicular to the radius from a fixed origin are λr and $\mu\theta$; find the path and show that the accelerations along and perpendicular to the radius vector are $\lambda^2 r-\frac{\mu^2\theta^2}{r}$ and $\mu\theta(\lambda+\frac{\mu}{r})$.

Solution:

Velocity component along the radius vector is $\dot{r}=\lambda r$ (1)

Velocity component along the transverse direction

$$r\dot{\theta}=\mu\theta \qquad (2)$$

From $\frac{(1)}{(2)}$ $\quad \frac{\dot{r}}{r\dot{\theta}}=\frac{\lambda r}{\mu\theta}$

$$\frac{1}{r}\frac{dr}{d\theta}=\frac{\lambda r}{\mu\theta}$$

$$\mu\,\frac{dr}{r^2}=\frac{\lambda}{\theta}\,d\theta$$

Integrating on both sides we set,

$$-\frac{\mu}{r} = \lambda \log\theta + c$$

Acceleration components along the radial direction

$$= \ddot{r} - r\dot{\theta}^2 \qquad (3)$$

From (1) $\Rightarrow$ $\ddot{r} = \lambda\dot{r} = \lambda(\lambda r) = \lambda^2 r$

From (2) $\Rightarrow$ $\dot{\theta} = \frac{\mu}{r}\theta$

Substituting in (3) we get

Acceleration component along the radius vector

$$= \lambda^2 r - \left(\frac{\mu}{r}\theta\right)^2 r$$

$$= \lambda^2 r - \frac{\mu^2\theta^2}{r}$$

Acceleration Component perpendicular to the radius vector

$$= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

$$= \frac{1}{r}\left[\frac{d}{dt}\left(rr\dot{\theta}\right)\right]$$

$$= \frac{1}{r}\left[\frac{d}{dt}(r\mu\theta))\right] \qquad \left[\because r\dot{\theta} = \mu\theta\right]$$

$$= \frac{1}{r}.\mu\left[\dot{r}\theta + r\dot{\theta}\right]$$

$$= \frac{\mu}{r}\left[\dot{r}\theta + \mu\theta\right]$$

$$= \frac{\mu\theta}{r}\left[\dot{r} + \mu\right]$$

Space for Hints

$$= \frac{\mu\theta}{r}[\lambda r + \mu] \qquad (\because \dot{r} = \lambda r)$$

$$= \mu\theta\left[\lambda + \frac{\mu}{r}\right]$$

Example: 9

If the angular velocity of a particle about a point in its plane of motion be constants, prove that the transverse component of its acceleration is proportional to the radial component of its velocity.

Solution:

Given that angular velocity of the particle about O is constant.

(ie) $\dot{\theta}$ = Constant = C (say) Now the transverse component of acceleration

$$= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) \quad = \frac{1}{r}C\;\frac{d}{dt}(r^2)$$

$$= \frac{C}{r}2r\dot{r} = 2C\dot{r}$$

(ie) Proportional to radial component of its velocity.

Exercises:

1) The velocities of a particle along and $\perp^r$ to the radius vector from a fixed origin are a and b. Find the path and the accelerations along and perpendicular to the radius.

2) A point P moves in a parabola in such a manner that the component velocity at right angles to the radius vector from the focus is constant vector from the focus is constant. Show that the acceleration of the point is constant in magnitude.

3) If a point moves so that its radial velocity is k times its transverse velocity, Show that its path is an equiangular spiral.

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4) If the radial and transverse velocities of a particle are always proportional to each other, show that the equation of the path is of the form $r = Ae^{k\theta}$ where A and K are constants.

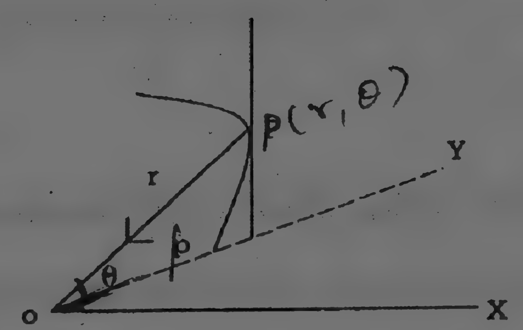
5) A particle describes the equiangular spiral $r = ae^{\theta \cot \alpha}$ whose pole is O. If its radial Velocity at distance r from o is $\frac{k}{r}$ (Constant), prove that the acceleration of the particle is $\frac{k \sec^2 \alpha}{r^3}$ towards O.

10.3. Differential Equation of Central orbit:

A particle of mass m moves in a plane under the action of a force m F directed towards a fixed point in its plane. Show that its angular momentum is constant and that the differential equation of its path is $\frac{d^2u}{d\theta^2}+u=F/h^2u.$ where u = 1/r, r be being the radius vector from the fixed point to the particle. θ is the vectorial angle and h, a constant.

Solution:

Let P be the position of the particle at any time t and let (r, θ) be the polar co – ordinates of P with respect to the fixed centre of force as pole. The acceleration of the particle along OP and Perpendicular to OP are $\ddot{r}-r\dot{\theta}$ and $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$. The only force acting on P is mF towards C. Therefore the equation of motion are



$$m(\ddot{r}-r\dot{\theta}^2)=-mF \text{ and} \tag{1}$$

$$m.\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})=0 \tag{2}$$

The second equation gives $r^2\dot{\theta}$ = a Constant say = $r^2\dot{\theta}=h$ (3)

Put $r=\frac{1}{u}$. *Then we have*

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$$\dot{r} = -\frac{1}{u^2} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = \frac{-1}{u^2}\frac{du}{d\theta}.\dot{\theta}$$

$$= -\frac{1}{u^2}\frac{du}{dt}\frac{h}{r^2}\text{[Using (3)]}$$

$$= -h\,\frac{du}{d\theta}. \qquad \left[\because r^2 = \frac{1}{u^2}\right]$$

Further, $\ddot{r} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right)$

$$= -h\,\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt} = -h\frac{d^2u}{d\theta^2}.\dot{\theta}$$

$$= -h\,\frac{d^2u}{d\theta^2}.\frac{h}{r^2} \qquad (\therefore \dot{\theta} = \frac{h}{r^2})$$

$$= -h^2u^2\,\frac{d^2u}{d\theta^2}$$

Substituting for $\ddot{r}$ and $\dot{\theta}$ in equation (1), we have

$$-h^2u^2\,\frac{d^2u}{d\theta^2} - r\left(\frac{h}{r^2}\right)^2 = -F$$

ie) $h^2u^2\,\dfrac{d^2u}{d\theta^2} + h^2u^3 = F$

ie, $\dfrac{d^2u}{d\theta^2} + u = \dfrac{F}{h^2u^2}.$

Which gives the differential equation of the central orbit in polar co – ordinates. The velocity components of the particle along OP and $\perp^r$ to OP are $\dot{r}$ and $r\dot{\theta}$ respectively and hence the linear moments in those directions are m $\dot{r}$ and $m\,r\,\dot{\theta}$ respectively. Taking moments of these vectors about O, we see that m $\dot{r}$ has no moment about O and $m\,r\,\dot{\theta}$ has moment given by $r\,(m\,r\,\dot{\theta}) = mr^2\,\dot{\theta}$.

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The moment of momentum (ie, the angular momentum) of the particle about O is $mr^2\dot{\theta}$. But $r^2\dot{\theta} = $ a constant $= h$. Hence the angular momentum of the particle about O is a constant = mh.

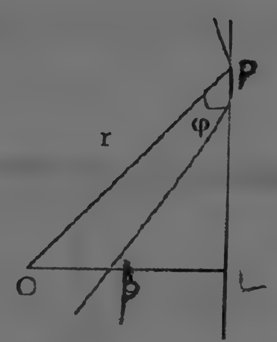
10.3.1 Perpendicular from the pole on the tangent Formulate in polar coordinates:

Let ϕ be the angle made by the tangent at P with he radius vector OP.

We know that $\tan\phi = r\dfrac{d\theta}{dr}$ (1)

From O draw OL $\perp^r$ to the tangent at P and let OL = p.

Then $\text{Sin}\,\phi = \dfrac{p}{r}$ or $p = r\,\text{Sin}\phi$ (2)



Let us eliminate ϕ between (1) and (2)

From (2), $\dfrac{1}{p^2} = \dfrac{1}{r^2\text{Sin}^2\phi} = \dfrac{1}{r^2}\text{Co}\sec^2\phi$

$= \dfrac{1}{r^2}(1+\text{Cot}^2\phi)$

$= \dfrac{1}{r^2}\left(1+\dfrac{1}{r^2}\left(\dfrac{dr}{d\theta}\right)^2\right)$ Substituting from (1)

(ie) $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^2}\left(\dfrac{dr}{d\theta}\right)^2$ (3)

Using $r = \dfrac{1}{u}, \dfrac{dr}{d\theta} = \dfrac{dr}{du}.\dfrac{du}{d\theta} = -\dfrac{1}{u^2}.\dfrac{du}{d\theta}$

Hence (3) becomes

$$\frac{1}{p^2} = u^2 + u^4 \cdot \frac{1}{r^4}\left(\frac{du}{d\theta}\right)^2 . \text{ (ie) } \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \qquad (4)$$

10.4 Pedal equation of the central orbit:

A particle P is describing a central orbit with a central acceleration F towards the pole O. If P be the length of the perpendicular from the pole on the tangent at the point P, Prove that $F = \frac{h^2 dp}{p^3 dr}$, h being the constant angular momentum of the particle about O.

Solution:

From the geometry of a plane curve, we have

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2$$

Taking $u = \frac{1}{r}$, *we have* $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$

Therefore, $\frac{1}{p^2} = u^2 + \left(\frac{dr}{d\theta}\right)^2.$

Differentiating both sides w.r.t.r we have

$$-\frac{2}{p^3}\frac{dp}{dr} = 2u\frac{du}{d\theta} + 2\frac{du}{dr}\cdot\frac{d}{dr}\left(\frac{du}{d\theta}\right)$$

$$= 2u\frac{du}{dr} + 2\frac{du}{dr}\cdot\frac{d^2u}{d\theta^2}\cdot\frac{d\theta}{dr}$$

$$= 2u\frac{du}{dr} + 2\frac{du}{dr}\cdot\frac{d^2u}{d\theta^2}.$$

$$= 2\frac{du}{dr}\left[\frac{d^2u}{d\theta^2} + u\right]$$

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$$= -\frac{2}{r^2}\left[\frac{d^2u}{d\theta^2}+u\right] \qquad\qquad \left(\because u=\frac{1}{r}\right)$$

But $\dfrac{d^2u}{d\theta^2}+u=\dfrac{F}{h^2u^2}$ where h is the constant angular momentum of the particle about O. Therefore

$$-\frac{2}{p^3}\frac{dp}{dr}=-\frac{2}{r^2}\frac{F}{h^2u^2}$$

ie, $\dfrac{h^2}{p^3}\dfrac{dp}{dr}=F.$

Problem: 1

Show that the velocity V at any point P (r,θ) in a central orbit is given by V = h/p, where h is the constant angular momentum of the particle about O.

Solution:

At the point P (r, θ), the particle has a velocity V whose components along the radius vector and perpendicular to the radius vector are $\dot{r}$ and $r\dot{\theta}$ respectively.

Therefore,

$$V^2 = r^2 + r^2\ \theta^2$$

But $r^2\ \dot{\theta} = h$ $\qquad\qquad \therefore \dot{\theta}=h/r^2$ and therefore

$$V^2 = r^2 + r^2(h/r^2)^2$$

$$= \dot{r}^2+\frac{h^2}{r^2}$$

But $\dot{r}=\dfrac{dr}{dt}=\dfrac{dr}{d\theta}.\dfrac{d\theta}{dt}=\dfrac{dr}{d\theta}.\dfrac{h}{r^2}$

$$V^2=\left\{\frac{h}{r^2}\left(\frac{dr}{d\theta}\right)\right\}^2+\frac{h^2}{r^2}$$

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$$= h^2\left\{\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right\} \qquad \cdots\cdots\cdots\cdots \qquad (1)$$

$$= h^2.\frac{1}{p^2} \text{ ie} V = h/p.$$

Note:

From (1), we have

$$V^2 = h^2\left[\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right]$$

$$= h^2\left[u^2+u^4\left(\frac{dr}{d\theta}\right)^2\right]$$

$u = 1/r$.

10.4.1 Pedal equation of some of the well – known curves:

(1) Circle – pole at any point:

Let C be the centre 'a' the radius, O the pole where OC = c.

Let P be any point on the circle and OL be the $\perp'$ from O on the tangent at P.

OP = r and OL = P.

From ΔOPC,

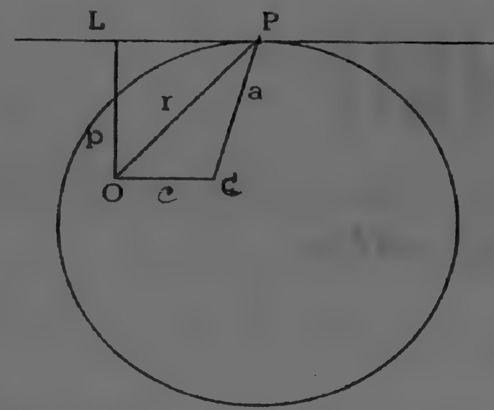
$c^2 = r^2 + a^2 - 2ra \text{ Cos } \angle OPC$

$= r^2 + a^2 - 2ra \text{ Cos } \angle POL$

$= r^2 + a^2 - 2ra.\ \frac{p}{r}$

$c^2 = r^2 + a^2 - 2ap$

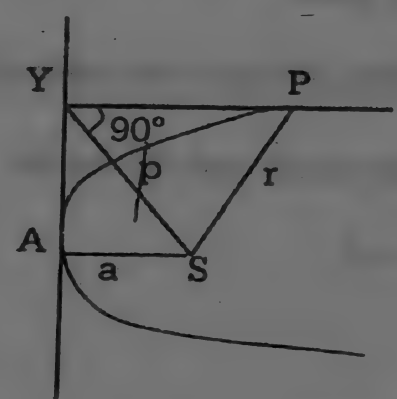
Space for Hints



Hence the pedal equations of the circle for a general position of the pole is $c^2 = r^2 + a^2 - 2ap$. When $c = a$, the pole is on the circumference and the equation is $r^2 = 2ap$.

(2) Parabola – Pole at focus:

To get the (p.r) equation to a parabola, we assume the geometrical property that if the tangent at P meets the tangent at the vertex A in Y and S is the focus, then SY, is $\perp^r$ to PY and the triangles SAY and SYP are similar.



Hence $\dfrac{SA}{SY} = \dfrac{SY}{SP}$

(ie) $\dfrac{a}{p} = \dfrac{p}{r}$ or $p^2 = ar$.

(3) Ellipse or Hyperbola pole at focus:

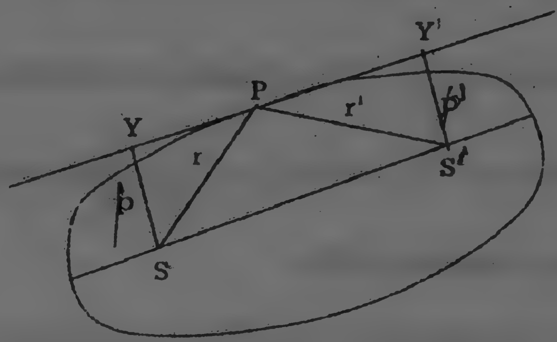
Let S and S' be the foci of the ellipse and SY, S' Y' be the $\perp^{rs}$ to the tangent at P. Taking S as the pole,

Let SP = r,

$S'P = r'$ $\quad SY = p.$ $\quad S'Y' = P'$

Let a and b be the semiaxes,

To find the (p.r) equation we assume the following geometrical properties of the ellipse.



i) $SP + S'P = 2a$

(ie) $r + r' = 2a$

ii) $SY.\ S'Y' = b^2$

(ie) $pp' = b^2$

iii) The tangent at P is equally inclined to the focal distances so that SPY and S'PY' are similar triangles.

So we have $\frac{p}{r} = \frac{p'}{r'}$

Now $\frac{b^2}{p^2} = \frac{pp'}{p^2}$ using (ii)

$= \frac{p'}{p} = \frac{r'}{r}$ using (iii)

$= \frac{2a - r}{r}$ using (i)

$= \frac{2a}{r} - 1$

Hence $\frac{b^2}{p^2} = \frac{2a}{r} - 1$ is the (p.r) equation to the ellipse.

By a similar argument, the (p,r) equation of the branch of the hyperbola nearer to the focus is $\frac{b^2}{p^2} = \frac{2a}{r} + 1.$

(4) Equiangular Spiral:

In any curve p = r Sin ϕ in the usual notation.

In the equilateral spiral, ϕ = constant = $\alpha\,(say)$

Hence p = r Sin α = Kr is the (p,r) equation to the sprial.

10.4.2 Velocities in a central orbit:

In every central orbit the areal velocity is constant and the linear velocity varies inversely as the $\perp^r$ from the centre upon the tangent to the path.

Let at time t the particle be at P (r, θ) and at time t + Δt, Let it it be at Q (r + Δr, θ + Δθ). Sectorial area OPQ described by the radius vector OP

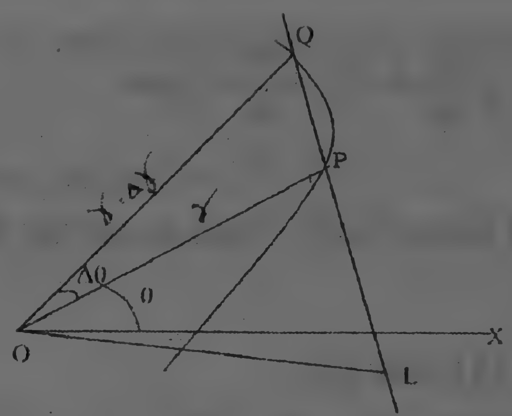
= Area of Δ OPQ nearly

$$= \frac{1}{2} OP\, OQ\, Sin\, \angle POQ$$

$$= \frac{1}{2} r\,(r + \Delta r)\, Sin\, \Delta\theta$$

$$= \frac{1}{2} r^2 \Delta\theta,$$ to the first order of smallness.

The rate of description of the area traced out by the radius vector joining the particle to a fixed point is called the areal velocity of the particle.



Hence in the central orbit, areal velocity of P

$$= \underset{\Delta \to 0}{Lt} \frac{1}{2} r^2 \frac{\Delta\theta}{\Delta t} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} h \qquad (1)$$

Since $r^2 \dot{\theta}$ = constant = h from equation (3) of Differential equation of central orbit.

Hence h = twice the areal velocity and as h is a constant, the areal velocity is constant. In other words, equal areas are described by the radius vector in equal times.

We can get another expression for the areal velocity.

Let Δs be the length of the arc PQ. Draw OL $\perp^r$ to PQ.

Sectorial area POQ = ΔPOQ nearly

$$= \frac{1}{2} PQ.OL$$

As $\Delta t \to O$, Q tends to coincide with P along the curve and the chord QP becomes the tangent at P. Length PQ = Δs nearly OL becomes the $\perp^r$ from O on the tangent at P. Let OL = p.

Hence areal velocity

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$$= \lim_{\Delta t \to 0} \frac{1}{2}\frac{\Delta s}{\Delta t}.p = \frac{1}{2}p\frac{ds}{dt} = \frac{1}{2}pv \qquad (2)$$

as $\frac{ds}{dt}$ is the rate of describing s and so in the linear velocity of p.

Hence combining (1) and (2).

$$\text{areal velocity} = \frac{1}{2}h = \frac{1}{2}pv.$$

Or $\quad h = pv$ (ie) $v = \frac{h}{p}$

$\therefore$ Hence linear velocity varies inversely as OP.

10.4.3 Two fold Problems in central orbits:

It is clear that two types of problems arise in connection with central orbits. They are

i) Given the orbit, to find the law of force to the pole.

ii) Given the law of force, to find path.

We shall first take up (1)

The different equation to the central orbit in polar co ordinates is

$$u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2}$$

Hence $P = h^2 u^2 \left(u + \frac{d^2u}{d\theta^2} \right)$

Since the orbit is given, u is known as a function of θ. Hence by differentiation, P can be got from the above equation.

In a few cases, we may know the (p,r) equation to the path. To find P we can use the equation $P = \frac{h^2}{p^3}.\frac{dp}{dr}$

Example: 1

Find the law of force towards the pole under which the curve $r^n = a^n \operatorname{Cos} n\theta$ can be described.

Solution:

Let $r^n = a^n \operatorname{Cos} n\theta$

Since $r = \frac{1}{u}$, the equation is $u^n a^n \operatorname{Cos} n\theta = 1$ (1)

Taking logarithms.

$$n \log u + n \log a + \log \operatorname{Cos} n\theta = 0 \quad (2)$$

Differentiating (2) with respect to θ.

$$n\frac{1}{u}\frac{du}{d\theta} - \frac{n \operatorname{Sin} n\theta}{\operatorname{Cos} n\theta} = 0$$

(ie) $\frac{du}{d\theta} = u \tan n\theta$ (3)

Differentiating (3) with respect to θ.

$$\frac{d^2u}{d\theta^2} = nu \sec^2 n\theta + \tan n\theta . \frac{du}{d\theta}$$

$$= nu \sec^2 n\theta + u \tan^2 n\theta [\because \text{using } (3)]$$

$$u + \frac{d^2u}{d\theta^2} = u + nu \sec^2 n\theta + u \tan^2 n\theta$$

$$= nu \sec^2 n\theta + u(1 + \tan^2 n\theta)$$

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$= nu \sec^2 n\theta + u \sec^2 n\theta$

$= (n + 1) u \sec^2 n\theta$

$= ((n+1)u.u^{2n}.a^{2n})$

[$\because$ using (1) to Substitute for $\sec^2 n\theta$]

$= (n + 1)a^{2n} u^{2n+1}$

$$P = h^2 u^2 \left(u + \frac{d^2u}{d\theta^2} \right) = h^2u^2(n+1)a^{2n}.u^{2n+1}$$

$= (n + 1) a^{2n} h^2 u^{2n+3} = (n + 1) a^{2n} h^2. \frac{1}{r^{2n+3}}$ ie) $P\alpha\frac{1}{r^{2n+3}}$ which means that the central acceleration varies inversely as the (2n + 3) rd Power of the distance.

Note:

From (4), P is positive only when $n + 1 > 0$.

(ie) $n > -1$

For values of $n < -1$, P will be negative and in such cases, the central forces will be a repulsive one. The above case is a comprehensive one giving the law of force for describing the following well known curves corresponding to particular values of n.

i) When $n = 1$, the equation is $r = a \cos \theta$. The curve is a circle and $P\alpha\frac{1}{r^7}$

ii) When $n = 2$, the equation is $r^2 = a^2 \cos 2\theta$. This is the Lemniscate of Bernowli and $P \alpha \frac{1}{r^7}$

iii) When $n = \frac{1}{2}$, the equation is $r^{1/2} = a^{1/2} \cos \theta/_2$.

(ie) $r = a \cos^2 \theta/_2 = \frac{a}{2}(1+\cos\theta)$

This is a cardioid and $P \alpha \frac{1}{r^4}$.

iv) When $n = -\frac{1}{2}$, the equation and is $r^{-1/2} = a^{-1/2} \cos \frac{\theta}{2}$

(ie) $a^{1/2} = r^{1/2} \cos \frac{\theta}{2}$.

So $r = \frac{a}{\cos^2 \theta/_2} = \frac{2a}{1+\cos\theta}$

(ie) $\frac{2a}{r} = 1+\cos\theta$

This is a parabola and $P \alpha \frac{1}{r^2}$

v) When $n = -2$, the equation is

$r^{-2} = a^{-2} \cos 2\theta$

(ie) $r^2 \cos 2\theta = a^2$.

This is a rectangular hyperbola. In this case the actual value of $P = a^{-4} h^2 r$.

The negative sign of P shows that the central force is a repulsive one.

Space for

Space for Hints

Example: 2

Find the law of force to an internal point under which a body will describe a circle.

Solution:

From pedal equation of some of the well – known curves the pedal equation of the circle for a general position of the pole is

$$c^2 = r^2 + a^2 - 2ap \qquad (1)$$

Differentiating with respect to r.

$$O = 2r - 2a\,\frac{dp}{dr}$$

$$\text{(ie)}\ \frac{dp}{dr} = \frac{r}{a}$$

Now the central acceleration.

$$P = \frac{h^2}{p^3}\cdot\frac{dp}{dr} = \frac{h^2 r}{p^3 a} = \frac{8h^2a^2.r}{(r^2+a^2-c^2)^3},$$ Substituting for p from (1)

Example: 3

A particle moves in an ellipse under a force which is always towards its focus. Find the law of force, the velocity at any point of the path and its periodic time.

Solution:

Referred to the focus as the focus as the pole, the polar equation to the ellipse is $\frac{\ell}{r} = 1 + e\cos\theta$ (1)

Where e is the eccentricity and ℓ is the semi – latus rectum.

From (1), $u = \frac{1}{r} = \frac{1+e\cos\theta}{\ell}$

Hence $\frac{du}{d\theta} = -\frac{e\sin\theta}{\ell}$ and $\frac{d^2u}{d\theta^2} = \frac{-e\cos\theta}{\ell}$

$u + \frac{d^2u}{d\theta} = \frac{1+e\cos\theta}{\ell} - \frac{e\cos\theta}{\ell} = \frac{1}{\ell}$

We know that $\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2} = \frac{1}{\ell}$

Hence $P = \frac{h^2u^2}{\ell} = \frac{\mu}{r^2}$ $where\ \mu = \frac{h^2}{\ell}$.

(ie) The force varies inversely as the square of the distance from the pole.

We have the result

$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$ [Refer $\perp^r$ from the pole on the tangent formula in polar coordinates]

$= \left(\frac{1+e\cos\theta}{\ell}\right)^2 + \left(\frac{e\sin\theta}{\ell}\right)^2$

$= \frac{1+2e\cos\theta+e^2}{\ell^2}$

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Also h= pv where v is the linear velocity.

Hence $v^2 = \frac{h^2}{p^2} = \frac{h^2(1+2e\cos\theta+e^2)}{\ell^2}$

$$= \frac{\mu\ell}{\ell^2}\left[1+e^2+2\left(\frac{\ell}{r}-1\right)\right] \qquad [\because \text{substituting for } e\cos\theta \text{ from (1)}]$$

$$= \frac{\mu}{\ell}\left(e^2+\frac{2\ell}{r}-1\right)$$

$$= \frac{\mu}{\ell}\left(\frac{2\ell}{r}-(1-e^2)\right)$$

$$= \mu\left(\frac{2}{r}-\frac{(1-e^2)}{\ell}\right) \qquad (2)$$

Now if a and b are the semi – axies of the ellipse,

We know that

$$\ell = \frac{b^2}{a} = \frac{a^2(1-e^2)}{a} = a(1-e^2).$$

Hence putting $\ell = a(1-e^2)$ in (2).

$v^2 = \mu\left[\frac{2}{r}-\frac{1}{a}\right]$ giving the velocity v.

Areal velocity in the orbit = $\frac{1}{2}$ h and this is constant.

The total area of the ellipse = π ab.

Periodic time T = $\frac{\pi ab}{(1/2h)} = \frac{2\pi ab}{h}$

$$= \frac{2\pi ab}{\sqrt{\mu\ell}}, \text{Since } \mu = \frac{h^2}{\ell}$$

$$= \frac{2\pi ab}{\sqrt{\mu}.b}.\sqrt{a}, \text{ Since } \ell = b^2/a$$

$$T = \frac{2\pi}{\sqrt{\mu}}.a^{3/2}$$

Example: 4

A particle moves in a curve under a central attraction so that its velocity at any point is equal to that in a circle at the same distance and under the same attraction. Show that the path is an equiangular spiral and that the law of force is that of the inverse cube.

Solution:

Let the central acceleration be P. If v is the velocity in a circle at a distance r under the normal acceleration P, then

$$\frac{v^2}{r} = P\,(ie)\, v^2 = \text{Pr} \qquad (1)$$

Since v is also the velocity in the central orbit, h = pv or v = $\frac{h}{p}$.

Putting this is (1), $\frac{h^2}{p^2} = \text{Pr}$

We know that

$$P = \frac{h^2}{p^3}.\frac{dp}{dr} \qquad (3)$$

Substituting (3) in (2)

$$\frac{h^2}{p^2} = \frac{h^2}{p^3}.\frac{dp}{dr}.r$$

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(ie) $\frac{dp}{p} = \frac{dr}{r}$

Integrating,

log p = log r + log A

(ie) p = Ar

(4) is clearly the (p,r) equation to an equiangular spiral.

From (4), $\frac{dp}{dr} = A$

Substituting this in (3),

$$P = \frac{h^2}{p^3}.A = \frac{Ah^2}{A^3r^3} \qquad [\because \text{using (4)}]$$

$$= \frac{h^2}{A^2}\left(\frac{1}{r^3}\right)$$

(ie)P $\alpha \frac{1}{r^3}$

Example: 5

A particle describes the curve $r^n = A \text{ Cos } n\theta + B \text{ Sin} n\theta$ under a force to the pole. Find the law of force.

Solution:

Given curve $r^n = A \text{ Con } n\ \theta + B \text{ Sin } n\theta$

Let $A = k \text{ Cos}\alpha$ and $B = k \text{ Sin } \alpha$

Where k and α are constants.

Then $r^n = k [\text{Cos } \alpha \text{ Cos } n\ \theta + \text{Sin } \alpha\ \text{ Sin } n\ \theta]$

Space for Hints

Put $r = \frac{1}{u}$

$r^n = u^{-n} = k \operatorname{Cos}(n\theta - \alpha)$ (1)

Taking log on both sides we get

$-n \log u = \log k + \log \operatorname{Cos}(n\theta - \alpha)$

Differentiating both sides w.r. to θ we get

$$\frac{-n}{u}\frac{du}{d\theta} = \frac{1}{\operatorname{Cos}(n\theta-\alpha)}(-\sin(n\theta-\alpha).n)$$

$$\frac{-n}{u}\frac{du}{d\theta} = -n\tan(n\theta - \alpha)$$

$$\frac{du}{d\theta} = u\tan(n\theta-\alpha)$$

Again differentiating w.r to θ we get

$$\frac{d^2u}{d\theta^2} = \frac{du}{d\theta}\tan(n\theta-\alpha) + un\operatorname{Sec}^2(n\theta-\alpha)$$

$$\frac{d^2u}{d\theta^2} = u\tan^2(n\theta-\alpha) + un\operatorname{Sec}^2(n\theta-\alpha) \qquad (2)$$

We know that, The differential equation of the path

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}. \qquad (3)$$

From (2) and (3) we get

$$P = h^2 u^2 [u + u\tan^2(n\theta-\alpha) + un\operatorname{Sec}^2(n\theta-\alpha)]$$

$$= h^2 u^2 [u(1 + \tan^2(n\theta-\alpha) + un\operatorname{Sec}^2(n\theta-\alpha]$$

$$= h^2u^2[u\operatorname{Sec}^2(n\theta-\alpha) + un\operatorname{Sec}^2(n\theta-\alpha)] = h^2u^3[(1+n)\sec^2(n\theta-\alpha)]$$

$$P = (1+n)\, h^2u^3 \operatorname{Sec}^2(n\theta - \alpha) \qquad (4)$$

Space for Hints

From (1), $r^n = k\,Cos\,(n\theta - \alpha)$

$$\frac{1}{u^n} = kCos(n\theta - \alpha)\left[\because r^n = \frac{1}{u^n}\right]$$

$$\frac{1}{Cos(n\theta - \alpha)} = ku^n$$

$$Sec(n\theta - \alpha) = ku^n \qquad (5)$$

From (4) and (5) we get

$$P = (1+n)\,h^2u^3(ku^n)^2$$

$$= (1+n)\,h^2u^3\,k^2\,u^{2n}$$

$$= (1+n)\,h^2\,k^2\,u^{2n+3}$$

$$= (1+n)\,h^2\,k^2\,\frac{1}{r^{2n+3}}(\because r = \frac{1}{u})$$

Thus $P \,\alpha \frac{1}{r^{2n+3}}$

ie) The force is inversely proportional to the $(2n+3)^{th}$ power of the distance from the pole.

Example: 6

A particle describes an equiangular spiral $r = ae^{\theta\cot\alpha}$ find the law of force.

Solution:

Given curve $r = ae^{\theta\cot\alpha}$ (1)

Put $r = \frac{1}{u}$

$$\frac{1}{u} = ae^{\theta\cot\alpha} \qquad (2)$$

Differentiating both sides w.r to θ we get

$$-\frac{1}{u^2}\frac{du}{d\theta} = a\cot\alpha.e^{\theta\cot\alpha}$$

$$= \frac{1}{u}\cot\alpha \qquad (\because \text{by}(2))$$

$$\Rightarrow \frac{du}{d\theta} = -u\,Cot\,\alpha \qquad (3)$$

Again differentiating both sides w.r.to θ we get

$$\Rightarrow \frac{d^2u}{d\theta^2} = -\cot\alpha\frac{du}{d\theta}$$

$$\Rightarrow \frac{d^2u}{d\theta^2} = u\,Cot^2\alpha \qquad (\because \text{by}(3)) \qquad (4)$$

We know that, The differential equation of the path is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2} \qquad (5)$$

From (4) and (5) we get.

$$\frac{P}{h^2u^2} = u + u\,Cot^2\alpha$$

$$\Rightarrow \frac{P}{h^2u^2} = u(1 + Cot^2\alpha)$$

$$\Rightarrow P = h^2u^3Cosec^2\,\alpha$$

Thus P $\alpha\frac{1}{r^3}$ ie) The force is inverse proportional to the 3rd power of the distance from the pole.

Space for Hints

Exercises:

1) Find the law of force towards the pole under which the following curves can be described

i) $r^2 = a^2 \text{Cos } 2\theta$ ii) $r^{1/2} = a^{1/2} \text{Cos } \frac{\theta}{2}$

iii) $r^n \text{Cos } \theta = a^n$ iv) $\frac{a}{r} = e^{n\theta}$

v) $a = r \text{ Sin } n\theta$ vi) $r = a \text{ Sin } n\theta$

vii) $r = a \cosh n\theta$.

2) Find the central acceleration under which the conic $\frac{\ell}{r} = 1 + e$ Cos θ can be described.

3) The velocity at any point of a central orbit is one half of what it would be for a circular orbit at the some distance obtain the law of forces.